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ALGEBRAIC CURVES IN $RP(1) \times RP(1)$

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ABSTRACT. We derive the analogs for curves in $RP(1) \times RP(1)$ of Fiedler's congruence for curves in $RP(2)$. We also prove a result on the complex orientations of dividing curves in $RP(1) \times RP(1)$.

INTRODUCTION

In $[M1, M2]$, S. Matsuoka considers the question: Which collections of curves in $RP(1) \times RP(1)$ can arise up to isotopy as a nonsingular real algebraic curve of fixed bidegree (d, r). In his sixteenth problem, Hilbert asked this same question about curves of fixed degree in $RP(2)$. Much progress has been made on Hilbert's question. See [V, W]. Matsuoka derived the analogs for curves in $RP(1) \times RP(1)$ of the following results for curves in $RP(2)$: Rokhlin's congruence (the Gudkov conjecture), the strong Petrovski inequalities, and the Arnold inequalities [V, (3.2)–(3.4), (3.6), (3.7)].

In this paper, we prove a result on the complex orientations of curves in $RP(1) \times RP(1)$ which seems to have no analog in $RP(2)$. We remark that Zvonilov has found some other restrictions on the complex orientations of dividing real algebraic curves on a general real algebraic surface [Z1, Z2]. These also apply to the situation considered here.

Finally we also derive the analogs of Fiedler's congruence [F]. Matsuoka's proof of the analog of Rokhlin's congruence, Fiedler's proof of his congruence, and our proof are all based on Marin's proof of Rokhlin's congruence [Ma]. Our argument shows how some of Matsuoka's argument may be simplified.

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1. COMPLEX ORIENTATIONS

A real algebraic curve RA of degree (d, r) is the zero locus in $RP(1) \times RP(1)$ of a real bihomogeneous polynomial $F(x_0, x_1; y_0, y_1)$ of degree (d, r). We assume RA has only real ordinary singular points. Thus RA is a collection of immersed closed curves with transverse intersections. Let CA denote the

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zero locus in $CP(1) \times CP(1)$. We say RA is dividing if CA modulo complex conjugation denoted $\widehat{CA}$ is homeomorphic to the space obtained by identifying a finite number of points on an orientable surface F with boundary. A choice of orientation for F (there are $2^{\beta_0(F)}$ choices) induces an orientation on RA. Such an orientation is called a complex orientation. This concept is due to Rokhlin [R] for nonsingular curves in $RP(2)$, and has been generalized to singular curves on a general real algebraic surface by Zonilov [Z2].

Recall for a nonsingular curve we have Harnack's inequality:

$$\beta_0(RA) \leq 1 + (d-1)(r-1).$$

A nonsingular curve with the maximal number of components is called an M-curve and must be dividing. In this case $\widehat{CA}$ is a punctured sphere. A curve is called an $(M-i)$-curve if it has i fewer components than an M-curve. If a nonsingular curve is dividing, the above inequality is equality modulo two.

Each factor of $RP(1) \times RP(1)$ is of course parallelizable. So we may take the product parallelization of $RP(1) \times RP(1)$ together with an arbitrary orientation. Then given an immersed oriented closed curve in $RP(1) \times RP(1)$, we may define the rotation number as the degree of the Gauss map. The rotation number $r(C)$ of a collection C of such curves is the sum of their individual rotation numbers. Note the rotation number of $\infty \times RP(1)$ and $RP(1) \times \infty$ is zero.

Theorem 1. *The rotation number of a dividing real algebraic curve in $RP(1) \times RP(1)$ with only real ordinary singularities with any of its complex orientations is zero.*

Proof. Let $A^+ \subseteq CA$ be the lift of $\widehat{CA}$ whose orientation at nonsingular points induced from the complex structure lifts the orientation on neighborhoods of the nonsingular points of $\widehat{CA}$. Recall [W, p. 58] that multiplication by i induces an orientation reversing isomorphism between the tangent bundle $T(RP(1) \times RP(1))$ and the normal bundle of $RP(1) \times RP(1)$ in $CP(1) \times CP(1)$. Moreover a tangent vector to RA in $RP(1) \times RP(1)$ gets sent to a vector tangent to CA and normal to $RP(1) \times RP(1)$. This is because CA intersected with a neighborhood of $RP(1) \times RP(1)$ is a smoothly immersed surface with the tangent plane to every point a complex line in the complex tangent bundle of the neighborhood. Let ν denote a small tubular neighborhood to $RP(1) \times RP(1)$ in $CP(1) \times CP(1)$. We can identify the pair $(\nu, \nu \cap CA)$ with the pair $(D(RP(1) \times RP(1)), DRA)$ by an orientation reversing map. Here D denotes the disk bundle associated to the tangent bundle. Note $\partial\nu \approx T^3$, the 3-torus and A^+ meets $\partial\nu$ in a link. Let $S = \partial(D(RP(1) \times RP(1)))$ thought of as the tangent circle bundle of $RP(1) \times RP(1)$. Note a point in S consists of a point of $RP(1) \times RP(1)$ together with an oriented tangent direction through that point. Thus to each point p on an oriented immersed curve γ in $RP(1) \times RP(1)$, we may associate the point in S given by the p and the oriented tangent to γ at p. Thus lying over a collection C of transverse immersed oriented curves in $RP(1) \times RP(1)$, there is an oriented link $L(C)$ in S. We see that

$L(RA)$ corresponds to $A^+ \cap \partial\nu$ under the above identification. Note $H_1(S)$ is isomorphic $H_1(RP(1) \times RP(1)) \oplus Z$ where the last Z is represented by the fiber and letting [ ] denote the homology class of, we have $[L(C)] = [C] \oplus r(C)$. Moreover the copy of $H_1(RP(1) \times RP(1))$ in $H_1(S)$ under this isomorphism dies under the map induced by inclusion in $H_1(CP(1) \times CP(1) - \mathrm{int}(\nu))$. This group is infinite cyclic generated by the fiber of $\partial\nu$ and $A^+ \cap \partial\nu$ bounds an oriented surface in $CP(1) \times CP(1) - \mathrm{int}(\nu)$, the theorem follows.

A component of a real algebraic curve which is the boundary of a disk in $RP(1) \times RP(1)$ is called an oval. Later we will refer to the disk as the interior of the oval. The other component of the complement is said to be exterior. Note that the rotation number of an essential simple closed curve is zero.

Corollary. *Let A be a nonsingular dividing real algebraic curve in $RP(1) \times RP(1)$ with one of its complex orientations. Then the number of ovals oriented one way is equal to the number of ovals oriented the other way.*

2. The analogs of Fiedler's congruence

In this section, we consider only nonsingular curves. By making a small change in the real coefficients of F which will only affect RA by a small isotopy, we may guarantee that CA is nonsingular. Note that the genus of CA is $(d-1)(r-1)$. The Harnack inequality mentioned above follows from this. The nonovals of RA are nonintersecting essential curves. Thus they share the same homology class up to sign: $\pm(s[\infty \times RP(1)] + t[RP(1) \times \infty])$ where s and t are relatively prime. Let l'' denote the number of nonovals. Let $R_1, R_2, \ldots, R_{l''}$ denote the closures of the components of the complement of the nonovals. Each R_i is an annulus. We index the R_i consecutively as one proceeds around the torus normal to the nonovals. We will also assume that d and r are even. In this case, the sign of F on $RP(1) \times RP(1) - RA$ is well defined. Replacing F by $-F$ if necessary, we insist that in the case $l'' = 0$ that $F \leq 0$ on the region exterior to every oval. If $l'' > 0$, we insist that $F \leq 0$ on the region of R_1 exterior to the ovals of R_1. Define B^+ to be the region of the torus where F is greater than or equal to zero, and B^- to be the region of the torus where F is less than or equal to zero. As the sign of F changes at each nonoval, we conclude that l'' is even. For dividing curves, define

$$l_+ = \sharp\{i \text{ even} \mid \text{the complex orientation on } \partial R_i \text{ does not extend to an orientation of } R_i\}.$$

In [M1], l_+ is denoted $\hat{l}$. Many of the above conventions and notations are also taken from Matsuoka. If we define l_- in the same way only restricting to i odd, then it is clear that $l_+ = l_- \bmod 2$. We will say that $B^\pm$ is even if each component of $B^\pm$ except possibly a single component in the exterior of all the ovals has even Euler characteristic.

Theorem 2. *Suppose RA is a nonsingular dividing curve of degree (d, r).*

(a) *If $l'' = 0$, $d \equiv r \equiv 0 \bmod 4$, RA is an M-curve, and B^- is even, then $\chi(B^+) \equiv 0 \bmod 16$.*

(b) *If l_+ is even (in particular if l'' is zero), $d \equiv r \equiv 0 \bmod 4$, and either B^+ or B^- is even, then $\chi(B^+) \equiv 0 \bmod 8$.*

(c) *If l_+ is odd, $d \equiv 2s \bmod 4$, $r \equiv 2t \bmod 4$, RA is an M-curve, and B^+ is even, then $\chi(B^+) + dr \equiv 0$ or $\pm 4 \bmod 16$.*

(d) *If l_+ is odd, $d \equiv 2s \bmod 4$, $r \equiv 2t \bmod 4$, RA is an M-curve, and B^- is even, then $\chi(B^+) - dr \equiv 0$ or $\pm 4 \bmod 16$.*

Proof. Let $W^\pm = B^\pm \cup A^+$. By Lemma (3.1) of [M1], we have that W^+ represents zero in $H_2(CP(1) \times CP(1), Z_2)$ in all the above cases. As $[W^+] + [W^-] = [RP(1) \times RP(1)] = 0$ in $H_2(CP(1) \times CP(1), Z_2)$, we have $[W^-] = 0$ as well. Since $CP(1) \times CP(1)$ is spin, both W^+ and W^- are characteristic surfaces. So we may consider the quadratic functions $q_\pm : H_1(W^\pm, Z_2) \to Z_4$ defined by Guillou and Marin. Let $\beta(q_\pm)$ in Z_8 denote the Brown invariant $[B]$ of $q_\pm$. According to Guillou and Marin's generalization of Rokhlin's Theorem [GM] we have

$$0 \equiv W^\pm \circ W^\pm + 2\beta(q_\pm) \mod 16.$$

Here $W^\pm \circ W^\pm$ denotes the number of double points counted with sign obtained when we push $W^\pm$ off itself. By patching together pushoffs of A^+ and $B^\pm$ (as in the proof of (2.4) [W], we have: $W^\pm \circ W^\pm = (1/2)CA \circ CA - \chi(B^\pm) = dr - \chi(B^\pm)$. As $\chi(B^+) + \chi(B^-) = \chi(RP(1) \times RP(1)) = 0$, we have:

$$\chi(B^+) \equiv dr + 2\beta(q_+) \equiv -dr - 2\beta(q_-) \mod 16.$$

As in [F, bottom of p. 567], the hypothesis that B^- is even guarantees that $q_+[\partial B_i^-] = 0$ where B_i^- is a connected component of B^- with $\chi(B_i^-)$ even. By definition this includes all the components of B^- except possibly one component exterior to all the ovals. Working outward from the innermost ovals we see that q_+ vanishes on every oval of RA. Moreover we see that q_+ takes the same fixed value on every nonoval of RA. In a similar way if B^+ is even, we see that q_- vanishes on the ovals of RA and takes the same fixed value on every nonoval. We need the following easily proved lemma.

Lemma 1. *Let γ be a simple closed curve on a closed surface F. Let q be a quadratic function on $H_1(F, Z_2)$ which vanishes on $[\gamma]$. Then γ is two-sided. Let F_γ result from surgery along γ and q_γ be the induced quadratic function on $H_1(F_\gamma, Z_2)$. Then $\beta(q) = \beta(q_\gamma)$.*

Under the hypothesis of (a), A^+ is a planar surface (as RA is an M-curve) and q_+ vanishes on each component of RA as l'' is zero. Thus W^+ is the union of two planar surfaces along their boundary. Thus if we surger W^+ along each component of RA, we will obtain a collection of 2-spheres. So by Lemma 1, $\beta(q_+)$ is zero. This proves case (a).

If RA is a $M-2i$ curve, W^+ surgered along the ovals of RA is the connected sum of $i+(1/2)l''-l_+$ tori and l_+ Klein bottles and perhaps some disjoint 2-spheres. On the other hand W^- surgered along the ovals of RA is $i+1$ tori and some 2-spheres if l'' is zero. It is the connected sum of $i+(1/2)l''-l_-$ tori and l_- Klein bottles and perhaps some disjoint 2-spheres otherwise. Moreover each of the Klein bottles and $(1/2)l''-l_+$ of the tori have a nonoval of RA representing a nontrivial two-sided homology class. Note the Brown invariant is additive on connected sums. The following lemma follows easily from the definition of β.

Lemma 2 (see [Ma, M1]). (a) *Let K be a Klein bottle, x a two-sided nonzero homology class in $H_1(K, Z_2)$, and q a quadratic function on $H_1(K, Z_2)$. If $q(x)$ is zero, then $\beta(q)$ is zero. If $q(x)$ is nonzero, then $\beta(q)$ is ± 2.*

(b) *Let T be a torus, and q a quadratic function on $H_1(T, Z_2)$, then $\beta(q)$ is zero* mod 4. *If x is a nonzero homology class in $H_1(T, Z_2)$, and $q(x)$ is zero, then $\beta(q)$ is zero.*

Now suppose B^- is even and l_+ is even. Then $\beta(q_+)$ is the same as the Brown invariant of the induced quadratic function for W^+ surgered along the ovals of RA. The contribution of each tori is zero mod 4. If q_+ (nonoval) is zero, the contribution of each of the l_+ Klein bottles is zero. If q_+ (nonoval) is not zero, the contribution of each of the l_+ Klein bottles is 2 mod 4. In either case the total contribution of the Klein bottles is zero mod 4. Thus $\beta(q_+)$ is zero mod 4. Similarly when B^+ is even, $\beta(q_-)$ is zero mod 4. This proves (b).

The proof of (c) proceeds in the same manner. Note in this case when RA is an M-curve every torus and Klein bottle has a nonoval. If q_- (nonoval) is zero, then $\beta(q_-)$ is zero. If q_- (nonoval) is nonzero, $\beta(q_-)$ is ± 2 mod 8. The proof of (d) is similar. Or one can derive (d) from (c) by rechoosing R_1 and the sign of F.

Remark. Our procedure for evaluating $\beta(q_\pm)$ using Lemmas 1 and 2 and considering $W^\pm$ surgered along the ovals of RA, provides a faster and more conceptual method than that used in the proof of the theorem of [M1]. These arguments could be substituted for some there.

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