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THE COMPLEXITY OF A MODULE AND ELEMENTARY ABELIAN SUBGROUPS: A GEOMETRIC APPROACH

PETER SYMONDS

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ABSTRACT. We present a proof of the theorem of Alperin and Evens that the complexity of a module is determined by the complexities of its restrictions to elementary abelian subgroups. We use only well-known properties of the spectral sequence.

The foundational material for the complexity theory of a module over a finite group can all be proved by elementary methods except for the key result that the complexity is determined by the complexities of the restrictions to the elementary abelian subgroups [AE]. The usual proofs use a theorem of Serre [S] and involve careful investigation of the behavior of products with respect to the filtration of the Lyndon-Hochschild-Serre spectral sequence.

We shall actually prove the following:

Theorem [AE]. *Let G be a finite group, k a field of characteristic p, and M a kG-module. Then $\mathrm{cx}_G(H^*(G; M)) = \max_J(\mathrm{cx}_G(H^*(J; M\downarrow_J^G)))$, where J runs through the abelian p-subgroups of G and cx_G denotes complexity.*

Remark. The usual form of this theorem as in [AE] is stronger in that J only runs through elementary abelian subgroups. However, the reduction from abelian to elementary abelian can be made as in [C], or by a simplified version of the proof in [AE]. I wish to thank the authors of [AE] for pointing out an error in the first version of this proof.

Proof. We may restrict to a Sylow p-subgroup of G without changing the complexity. So we assume that G is a p-group. Then $\mathrm{cx}_G(M) = \gamma(H^*(G; M))$, where γ denotes the growth function. We now use induction on $|G|$. If G is abelian then there is nothing to prove, so we may assume that G has an irreducible complex representation, V for example, of dimension $n \geq 2$. Let $\mathbb{P}(V)$ denote the projective space of V and triangulate it in such a way that G acts admissibly. The stabilizers of the cells are all proper subgroups.

Now consider the fibration:

$$\mathbb{P}(V) \xrightarrow{i} EG \times_G \mathbb{P}(V) \to BG.$$

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This leads to the spectral sequence for equivariant cohomology:

$$E_2^{**} = H^*(G; H^*(\mathbb{P}(V); M)) \Rightarrow H_G^*(\mathbb{P}(V); M).$$

The action of G on $H^*(\mathbb{P}(V); \mathbb{Z})$ is trivial, since it factors through an action of the unitary group, which is connected. Thus the E_2 page can be identified with $H^*(G; M) \otimes H^*(\mathbb{P}(V); k)$. The canonical line bundle l over $\mathbb{P}(V)$ extends to a line bundle l' over $EG \times_G \mathbb{P}(V)$. As $l = i^* l'$, we obtain $c_1(l) = i^* c_1(l')$, where c_1 denotes the first Chern class. Consider the diagram

$$\begin{array}{ccc} H_G^0(\mathbb{P}(V); M) \otimes H_G^*(\mathbb{P}(V); k) & \xrightarrow{i^*} & M \otimes H^*(\mathbb{P}(V); k) \\ \downarrow & & \wr\| \\ E_\infty^{*,0} = H_G^*(\mathbb{P}(V); M) & \xrightarrow{i^*} & H^*(\mathbb{P}(V); M) = E_2^{*,0}; \end{array}$$

the upper horizontal arrow is surjective and therefore, so is the lower one. This forces the spectral sequence to collapse by the Leray-Hirsch Theorem [D, H], and so

$$H_G^s(\mathbb{P}(V); M) \cong \bigoplus_{i=0}^{n} H^{s-2i}(G; M),$$

(the nonzero terms are contained in a horizontal strip of width $2n-1$). Hence $\gamma(H_G^*(\mathbb{P}(V); M)) = \gamma(H^*(G; M))$.

We can also consider the spectral sequence on cells [B, p. 173]:

$$E_1^{p,q} = \bigoplus_{\substack{\sigma \in G\backslash\mathbb{P}(V) \\ \dim\sigma = p}} H^q(\mathrm{stab}_G\,\sigma; M) \Rightarrow H_G^{p+q}(\mathbb{P}(V); M).$$

The succeeding pages of the spectral sequence can only have smaller groups as entries, so

$$\dim H_G^s(\mathbb{P}(V); M) \le \sum_{i=0}^{2(n-1)} \sum_{\dim\sigma = i} \dim H^{s-i}(\mathrm{stab}_G\,\sigma; M),$$

(the nonzero terms lie within a vertical strip.) Hence $\gamma(H_G^*(\mathbb{P}(V); M)) \le \max_\sigma(\gamma(H^*(\mathrm{stab}_G\,\sigma; M)))$.

All the stabilizers are proper subgroups and the inequality in the other direction is trivial, hence the proof is complete by induction.

References

[AE] J. L. Alperin and L. Evens, *Representations, resolutions and Quillen's dimension theorem*, J. Pure Appl. Algebra **26** (1981), 1–9.

[B] K. S. Brown, *Cohomology of groups*, Springer-Verlag, New York, 1982.

[C] J. F. Carlson, *Restrictions of modules over modular group algebras*, J. Algebra **53** (1978), 334–343.

[D] A. Dold, *Relations between ordinary and extraordinary homology*, Algebraic Topology, Student's Guide (J. F. Adams, ed.), London Math. Soc. Lecture Note Ser., No. 4, Cambridge Univ. Press, London and New York, 1972.

[H] D. Husemoller, *Fibre bundles* 2nd ed., Springer-Verlag, New York, 1975.

[S] J.-P. Serre, *Sur la dimension cohomologique des groupes profinis*, Topology **3** (1965), 413–420.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA T6G 2E1, CANADA