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ESTIMATES FOR THE MAXIMAL OPERATOR OF THE ORNSTEIN-UHLENBECK SEMIGROUP

CRISTIAN E. GUTIÉRREZ AND WILFREDO O. URBINA

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ABSTRACT. We show pointwise estimates for the maximal operator of the Ornstein-Uhlenbeck semigroup for functions that are integrable with respect to the Gaussian measure. The estimates are used to prove pointwise convergence.

The Ornstein-Uhlenbeck semigroup is defined by

$$T_t f(x) = \int_{R^n} k(t, x, y) f(y)\, dy,$$

where

$$k(t, x, y) = \pi^{-n/2}(1 - e^{-2t})^{-n/2} \exp\left(-\frac{|e^{-t}x - y|^2}{1 - e^{-2t}}\right), \qquad t > 0, \ x \in R^n.$$

This family of operators was considered by L. S. Ornstein and G. E. Uhlenbeck to construct a theory of Brownian motion [N]. The infinitesimal generator of this semigroup is $L = \frac{1}{2}\Delta - x \cdot \nabla$, and the eigenfunctions are the Hermite polynomials. If $0 < r < 1$ then the Poisson-Hermite integral of the function f is given by

$$P_r f(x) = T_{\log \frac{1}{r}} f(x).$$

For $x \in R^n$ we set $\gamma(x) = \pi^{-n/2} e^{-|x|^2}$, and given $p \geq 1$, by $L^p(R^n, \gamma)$ we denote the class of functions that are integrable to the pth power over R^n with respect to the measure $\gamma(x)\, dx$. Also, $\|\cdot\|_{p,\gamma}$ denotes the usual norm. Let

$$P^* f(x) = \sup_{0<r<1} |P_r f(x)|.$$

In one dimension the maximal operator P^* was studied by B. Muckenhoupt [M] in 1969 and, independently, by C. P. Calderón [C] also in 1969. They showed that P^* is bounded in $L^p(R, \gamma)$ if $p > 1$ and is of weak-type $(1, 1)$ with respect to γ. C. P. Calderón [C], by adapting the well-known results of Abel and Cesàro summability of the multiple Fourier series to the case of Abel

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summability of multiple Hermite series, (see [Z]) has shown the boundedness in $L^p(R^n, \gamma)$, $p > 1$, of P^* with respect to the Gaussian measure.

This paper concerns the behavior of P^* in $L^1(R^n, \gamma)$. We introduce the centered maximal function defined by

$$M_\gamma f(x) = \sup_{r>0} \frac{1}{\gamma(B_r(x))} \int_{B_r(x)} |f(y)|\gamma(y)\, dy,$$

$B_r(x)$ is the ball in R^n with center x and radius r, and $\gamma(B_r(x)) = \int_{B_r(x)} \gamma(t)\, dt$. The standard proof using the Besicovitch covering lemma shows that M_γ is of weak-type $(1, 1)$ with respect to the measure $\gamma\, dx$.

In this paper we show the following:

Theorem. *If $f \in L^1(R^n, \gamma)$ then the following pointwise estimate holds:*

$$P^* f(x) \le c_n M_\gamma f(x) + (2 \vee |x|)^n e^{|x|^2} \|f\|_{1,\gamma}.$$

Also $P_r f \to f$ as $r \to 1^-$ in $L^1(R^n, \gamma)$ and for almost every x we have

$$P_{\sqrt{1-s}} f(y/\sqrt{1-s}) \to f(x),$$

as $s \to 0$ for $|x - y| \le s^{1/2}$.

In one dimension our theorem gives a new proof of the weak-type $(1, 1)$ of P^* proved in [C, M]. This follows because the function $(2 \vee |x|)e^{|x|^2}$ belongs to weak $L^1(R, \gamma)$; see (5) below.

Proof of the theorem. We may assume $f \ge 0$. If we set

$$u(y, s) = s^{-n/2} \int_{R^n} \gamma\left(\frac{y - z}{s^{1/2}}\right) f(z)\, dz,$$

then

$$P^* f(x) = \sup_{0<t<1} u(\sqrt{1-t}\,x, t).$$

Let

$$P_1^* f(x) = \sup_{0<t<1/|x|^2 \wedge 1/4} u(\sqrt{1-t}\,x, t)$$

and

$$P_2^* f(x) = \sup_{1/|x|^2 \wedge 1/4 \le t<1} u(\sqrt{1-t}\,x, t).$$

We introduce the maximal function

$$Mf(x) = \sup_{(y,s)\in\Gamma(x)} u(y, s),$$

where $\Gamma(x) = \{(x, s) : |x - y| \le s^{1/2},\ 0 < s < 1/|x|^2 \wedge 1/4\}$.

We shall show that

$$P_1^* f(x) \le Mf(x) \le c_n M_\gamma f(x) \tag{1}$$

and

$$P_2^* f(x) \le (2 \vee |x|)^n e^{|x|^2} \|f\|_{1,\gamma}. \tag{2}$$

The first inequality in (1) follows because if $0 < t < 1/|x|^2 \wedge 1/4$ then $(\sqrt{1-t}x, t) \in \Gamma(x)$.

Now set $a_j = j^{1/2}$ for $j = 1, 2, \dots$. We write

$$u(y, s) = s^{-n/2} \sum_{j=1}^{\infty} \int_{a_{j-1}s^{1/2} \le |y-z| < a_j s^{1/2}} \gamma\left(\frac{y-z}{s^{1/2}}\right) f(z)\, dz.$$

If $(y, s) \in \Gamma(x)$ and $|y-z| < a_j s^{1/2}$ then $|x-z| < (1+a_j)s^{1/2}$ and consequently,

$$u(y, s) \le s^{-n/2} \sum_{j=1}^{\infty} \gamma(a_{j-1}) \int_{|x-z|<(1+a_j)s^{1/2}} f(z)\, dz.$$

We have $|z|^2 = |z-x|^2 + 2x \cdot (z-x) + |x|^2$, hence

$$\begin{aligned}
&\int_{|x-z|<(1+a_j)s^{1/2}} f(z)\, dz \\
&\quad \le \exp(|x|^2 + 2|x|(1+a_j)s^{1/2} + (1+a_j)^2 s) \int_{|x-z|<(1+a_j)s^{1/2}} f(z) e^{-|z|^2}\, dz \\
&\quad \le \exp(|x|^2 + 2|x|(1+a_j)s^{1/2} + (1+a_j)^2 s) M_\gamma f(x) \int_{|x-z|<(1+a_j)s^{1/2}} e^{-|z|^2}\, dz \\
&\quad = \exp(2|x|(1+a_j)s^{1/2} + (1+a_j)^2 s) M_\gamma f(x) \\
&\qquad \times \int_{|x-z|<(1+a_j)s^{1/2}} e^{-|z-x|^2 - 2x\cdot(z-x)}\, dz \\
&\quad \le \exp(4|x|(1+a_j)s^{1/2} + (1+a_j)^2 s) M_\gamma f(x) \int_{|z|<(1+a_j)s^{1/2}} e^{-|z|^2}\, dz \\
&\quad \le \Omega_n (1+a_j)^n s^{n/2} \exp(4|x|(1+a_j)s^{1/2} + (1+a_j)^2 s) M_\gamma f(x).
\end{aligned}$$

By adding in j we get for $(y, s) \in \Gamma(x)$

$$u(y, s) \le c_n M_\gamma f(x),$$

and then the second inequality in (1) follows.

To prove (2) observe that if $0 < t < s$ and $x, y \in R^n$ then for any $z \in R^n$ we have

$$\frac{|y-z|^2}{s} - \frac{|x-z|^2}{t} \le \frac{|y-x|^2}{s-t}.$$

Thus

$$u(y, t) \le u(x, s) \left(\frac{s}{t}\right)^{n/2} \exp\left(\frac{|x-y|^2}{s-t}\right),$$

for $0 < t < s$. In particular

$$u(\sqrt{1-t}x, t) \le u(0, 1)(1/t)^{n/2} e^{|x|^2}$$

for $0 < t < 1$. If $t \geq 1/|x|^2 \wedge 1/4$ then $1/t \leq (2 \vee |x|)^2$ and hence (2) follows.

The convergence in L^1 is proved in the same way as in the proof of Theorem 2 of [M].

To show the a.e. convergence define

$$\Omega f(x) = \lim_{\alpha \to 0} \left(\sup_{(y,s) \in \Gamma(x),\ 0<s<\alpha} |u(y,s) - f(x)| \right),$$

and set

$$f = f\phi_k + f(1-\phi_k) = f_1 + f_2,$$

where k is an integer greater than 1, and ϕ_k is the characteristic function of the ball $B_k(0)$. We shall show that

(3) $$\Omega f(x) \leq c_n M_\gamma f_2(x)$$

for every $k > 1$ and almost every x with $|x| \leq k-1$.

We have that

(4) $$\lim_{r \to 0} \frac{1}{\gamma(B_r(x))} \int_{B_r(x)} |f(z) - f(x)| e^{-|z|^2}\, dz = 0,$$

for almost every x. Let x be a point satisfying (4), then for every $\varepsilon > 0$ there exists δ, $0 < \delta < 1$ such that

$$\frac{1}{\gamma(B_r(x))} \int_{B_r(x)} |f(z) - f(x)| e^{-|z|^2}\, dz < \varepsilon$$

for $0 < r < \delta$. Set

$$g(z) = \begin{cases} f(z) - f(x), & \text{if } |x-z| \leq \delta, \\ 0, & \text{if } |x-z| > \delta, \end{cases}$$

and notice that $M_\gamma g(x) < \varepsilon$. If we set $u_i(y,s) = \gamma_{s^{1/2}} * f_i(y)$, $i = 1, 2$ then we have

$$u(y,s) - f(x) = u_1(y,s) - f_1(x) + u_2(y,s) - f_2(x).$$

We write

$$\begin{aligned} u_1(y,s) - f_1(x) &= \pi^{-n/2} s^{-n/2} \int_{R^n} \gamma\left(\frac{y-z}{s^{1/2}}\right)(f_1(z) - f_1(x))\, dz \\ &= \pi^{-n/2} s^{-n/2} \left(\int_{|x-z|\leq\delta} + \int_{|x-z|>\delta} \right) \\ &= I + II. \end{aligned}$$

If $|x| \leq k-1$ and $(y,s) \in \Gamma(x)$ then by (1) we have

$$|I| \leq Mg(x) \leq c_n M_\gamma g(x) < c_n \varepsilon.$$

To estimate II, observe that if $|x-y| \leq s^{1/2}$ and $\delta \geq 2s^{1/2}$ then

$$\begin{aligned} |II| &\leq \pi^{-n/2} s^{-n/2} \int_{|y-z|\geq\delta/2} e^{-|y-z|^2/s} |f_1(z)|\, dz \\ &\quad + \pi^{-n/2} s^{-n/2} \int_{|x-z|>\delta} e^{-|y-z|^2/s} |f_1(x)|\, dz \\ &= II_1 + II_2. \end{aligned}$$

We have

$$II_1 = \pi^{-n/2}s^{-n/2}\int_{\{|y-z|\geq\delta/2,\ |z|\leq k\}} \exp\left(-\frac{|y-z|^2}{s}+|z|^2\right)|f(z)|e^{-|z|^2}\,dz$$
$$\leq \pi^{-n/2}s^{-n/2}\exp((-\delta^2/4s)+k^2)\|f\|_{1,\gamma},$$

and

$$II_2 = |f_1(x)|\pi^{-n/2}s^{-n/2}\int_{|x-y+t|\geq\delta} e^{-|t|^2/s}\,dt$$
$$\leq |f_1(x)|\pi^{-n/2}s^{-n/2}\int_{|t|\geq\delta/2} e^{-|t|^2/s}\,dt$$
$$= |f_1(x)|\pi^{-n/2}\int_{|x|\geq\delta/2s^{1/2}} e^{-|x|^2}\,dx.$$

If $|x|\leq k$ then $f_2(x)=0$ and consequently by (1)

$$|u_2(y,s)-f_2(x)|\leq c_n M_\gamma f_2(x)$$

for $(y,s)\in\Gamma(x)$.

Therefore, collecting estimates and taking supremum over $(y,s)\in\Gamma(x)$, $0<s<\alpha$, and letting $\alpha\to 0$ we obtain

$$\Omega f(x)\leq c_n(\varepsilon+M_\gamma f_2(x)),$$

for every $\varepsilon>0$, and a.e. x with $|x|\leq k-1$, which gives (3). Given $\varepsilon>0$, we pick k such that $\|f_2\|_{1,\gamma}<c\varepsilon^2$ then by (3) and the weak-type $(1,1)$ of M_γ we obtain

$$\gamma\{x:|x|\leq k-1,\ \Omega f(x)>\varepsilon\}<\varepsilon,$$

which implies $\Omega f(x)=0$ almost everywhere.

Remarks. (A) Set $\varphi(x)=(2\vee|x|)^n e^{|x|^2}$ and $E_\lambda=\{x:\varphi(x)>\lambda\}$, $\lambda>0$. We shall show that

$$\gamma(E_\lambda)\leq c(1+\ln^+\lambda)^{n-1}/\lambda. \tag{5}$$

Since γ is a finite measure, it is enough to prove (5) for λ large. The function φ increases with $|x|$ then $E_\lambda=\{x:|x|\geq r_0\}$ where $r_0^n\exp(r_0^2)=\lambda$. Therefore $r_0\sim(\ln\lambda)^{1/2}$ for λ large. Hence

$$\gamma(E_\lambda)=c_n\int_{r_0}^\infty r^{n-1}e^{-r^2}\,dr$$
$$=c_n e^{-r_0^2}\int_0^\infty (t+r_0^2)^{(n/2)-1}e^{-t}\,dt$$
$$=c_n e^{-r_0^2}r_0^{n-2}\int_0^\infty (1+t/r_0^2)^{(n/2)-1}e^{-t}\,dt$$
$$\sim c_n e^{-r_0^2}r_0^{n-2}=c_n r_0^{2(n-1)}/\lambda,$$

and (5) follows.

(B) The theorem implies that $P_r(x) \to f(x)$ as $r \to 1^-$ a.e. This last result can also be obtained as a consequence of Lemma (2.17) of [C]. The one-dimensional case is due to E. Hille [H].

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Department of Mathematics, Temple University, Philadelphia, Pennsylvania 19122

Departamento de Matemática, Universidad Central de Venezuela, Caracas, Venezuela