ESTIMATES FOR THE MAXIMAL OPERATOR OF THE ORNSTEIN-UHLENBECK SEMIGROUP

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Abstract. We show pointwise estimates for the maximal operator of the Ornstein-Uhlenbeck semigroup for functions that are integrable with respect to the Gaussian measure. The estimates are used to prove pointwise convergence.

The Ornstein-Uhlenbeck semigroup is defined by

$$T_t f(x) = \int_{\mathbb{R}^n} k(t, x, y) f(y) dy,$$

where

$$k(t, x, y) = \pi^{-n/2} (1 - e^{-2t})^{-n/2} \exp \left( -\frac{|e^{-t}x - y|^2}{1 - e^{-2t}} \right), \quad t > 0, \ x \in \mathbb{R}^n.$$

This family of operators was considered by L. S. Ornstein and G. E. Uhlenbeck to construct a theory of Brownian motion [N]. The infinitesimal generator of this semigroup is $L = \frac{1}{2} \Delta - x \cdot \nabla$, and the eigenfunctions are the Hermite polynomials. If $0 < r < 1$ then the Poisson-Hermite integral of the function $f$ is given by

$$P_r f(x) = T_{\log \frac{1}{r}} f(x).$$

For $x \in \mathbb{R}^n$ we set $\gamma(x) = \pi^{-n/2} e^{-|x|^2}$, and given $p \geq 1$, by $L^p(\mathbb{R}^n, \gamma)$ we denote the class of functions that are integrable to the $p$th power over $\mathbb{R}^n$ with respect to the measure $\gamma(x) \, dx$. Also, $\| \cdot \|_{p, \gamma}$ denotes the usual norm. Let

$$P^* f(x) = \sup_{0 < r < 1} |P_r f(x)|.$$

In one dimension the maximal operator $P^*$ was studied by B. Muckenhoupt [M] in 1969 and, independently, by C. P. Calderón [C] also in 1969. They showed that $P^*$ is bounded in $L^p(R, \gamma)$ if $p > 1$ and is of weak-type $(1, 1)$ with respect to $\gamma$. C. P. Calderón [C], by adapting the well-known results of Abel and Cesàro summability of the multiple Fourier series to the case of Abel

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summability of multiple Hermite series, (see [Z]) has shown the boundedness in $L^p(R^n, \gamma)$, $p > 1$, of $P^*$ with respect to the Gaussian measure.

This paper concerns the behavior of $P^*$ in $L^1(R^n, \gamma)$. We introduce the centered maximal function defined by

$$M^*_\gamma f(x) = \sup_{r > 0} \frac{1}{\gamma(B_r(x))} \int_{B_r(x)} |f(y)||\gamma(y)| dy,$$

$B_r(x)$ is the ball in $R^n$ with center $x$ and radius $r$, and $\gamma(B_r(x)) = \int_{B_r(x)} \gamma(t) dt$. The standard proof using the Besicovitch covering lemma shows that $M^*_\gamma$ is of weak-type $(1, 1)$ with respect to the measure $\gamma dx$.

In this paper we show the following:

**Theorem.** If $f \in L^1(R^n, \gamma)$ then the following pointwise estimate holds:

$$P^*_*f(x) \leq c_n M^*_\gamma f(x) + (2 \vee |x|)^n e^{\ln^2} \|f\|_{1, \gamma}.$$

Also $P_r f \to f$ as $r \to 1^-$ in $L^1(R^n, \gamma)$ and for almost every $x$ we have

$$P^* \sqrt{1-s} f(y/\sqrt{1-s}) \to f(x),$$

as $s \to 0$ for $|x-y| \leq s^{1/2}$.

In one dimension our theorem gives a new proof of the weak-type $(1, 1)$ of $P^*$ proved in [C, M]. This follows because the function $(2 \vee |x|)^n e^{\ln^2}$ belongs to weak $L^1(R, \gamma)$; see (5) below.

**Proof of the theorem.** We may assume $f \geq 0$. If we set

$$u(y, s) = s^{-n/2} \int_{R^n} \gamma \left( \frac{y - z}{s^{1/2}} \right) f(z) dz,$$

then

$$P^*_* f(x) = \sup_{0 < t < 1} u(\sqrt{1-tx}, t).$$

Let

$$P^*_1 f(x) = \sup_{0 < t < 1/|x|^2 \wedge 1/4} u(\sqrt{1-tx}, t)$$

and

$$P^*_2 f(x) = \sup_{1/|x|^2 \wedge 1/4 \leq t < 1} u(\sqrt{1-tx}, t).$$

We introduce the maximal function

$$M f(x) = \sup_{(x, s) \in \Gamma(x)} u(x, s),$$

where $\Gamma(x) = \{(x, s) : |x - y| \leq s^{1/2}, 0 < s < 1/|x|^2 \wedge 1/4\}$.

We shall show that

$$P^*_1 f(x) \leq M f(x) \leq c_n M^*_\gamma f(x) \leq c_n M^*_\gamma f(x).$$
and

\[ P^*_2 f(x) \leq (2 \vee |x|)^n e^{|x|^2} \|f\|_1 \gamma. \]

The first inequality in (1) follows because if \(0 < t < 1/|x|^2 \wedge 1/4\) then \((\sqrt{1 - tx}, t) \in \Gamma(x)\).

Now set \(a_j = j^{1/2}\) for \(j = 1, 2, \ldots\). We write

\[
u(y, s) = s^{-n/2} \sum_{j=1}^{\infty} \int_{a_{j-1}s^{1/2} \leq |y - z| < a_j s^{1/2}} \gamma \left( \frac{y - z}{s^{1/2}} \right) f(z) \, dz.
\]

If \((y, s) \in \Gamma(x)\) and \(|y - z| < a_j s^{1/2}\) then \(|x - z| < (1 + a_j)s^{1/2}\) and consequently,

\[
u(y, s) \leq s^{-n/2} \sum_{j=1}^{\infty} \gamma(a_{j-1}) \int_{|x - z| < (1 + a_j)s^{1/2}} f(z) \, dz.
\]

We have \(|z|^2 = |z - x|^2 + 2x \cdot (z - x) + |x|^2\), hence

\[
\int_{|x - z| < (1 + a_j)s^{1/2}} f(z) \, dz
\leq \exp(|x|^2 + 2|x|(1 + a_j)s^{1/2} + (1 + a_j)^2 s) \int_{|x - z| < (1 + a_j)s^{1/2}} f(z) e^{-|z|^2} \, dz
\leq \exp(|x|^2 + 2|x|(1 + a_j)s^{1/2} + (1 + a_j)^2 s) M_\gamma f(x) \int_{|x - z| < (1 + a_j)s^{1/2}} e^{-|z|^2} \, dz
\leq \exp(4|x|(1 + a_j)s^{1/2} + (1 + a_j)^2 s) M_\gamma f(x)
\times \int_{|x - z| < (1 + a_j)s^{1/2}} e^{-|z - x|^2 - 2x \cdot (z - x)} \, dz
\leq \Omega_n (1 + a_j)^n s^{n/2} \exp(4|x|(1 + a_j)s^{1/2} + (1 + a_j)^2 s) M_\gamma f(x).
\]

By adding in \(j\) we get for \((y, s) \in \Gamma(x)\)

\[
u(y, s) \leq c_n M_\gamma f(x),
\]

and then the second inequality in (1) follows.

To prove (2) observe that if \(0 < t < s\) and \(x, y \in \mathbb{R}^n\) then for any \(z \in \mathbb{R}^n\) we have

\[
\frac{|y - z|^2}{s} - \frac{|x - z|^2}{t} \leq \frac{|y - x|^2}{s - t}.
\]

Thus

\[
u(y, t) \leq \nu(x, s) \left( \frac{s}{t} \right)^{n/2} \exp \left( \frac{|x - y|^2}{s - t} \right),
\]

for \(0 < t < s\). In particular

\[
u(\sqrt{1 - tx}, t) \leq \nu(0, 1)(1/t)^{n/2} e^{|x|^2}.
\]
for $0 < t < 1$. If $t \geq 1/|x|^2 \wedge 1/4$ then $1/t \leq (2 \vee |x|)^2$ and hence (2) follows.

The convergence in $L^1$ is proved in the same way as in the proof of Theorem 2 of [M].

To show the a.e. convergence define

$$\Omega f(x) = \lim_{\alpha \to 0} \left( \sup_{(y,s) \in \Gamma(x), \ 0 < s < \alpha} |u(y,s) - f(x)| \right),$$

and set

$$f = f \phi_k + f(1 - \phi_k) = f_1 + f_2,$$

where $k$ is an integer greater than 1, and $\phi_k$ is the characteristic function of the ball $B_k(0)$. We shall show that

$$\Omega f(x) \leq c_n M \gamma f_2(x)$$

for every $k > 1$ and almost every $x$ with $|x| < k - 1$.

We have that

$$\lim_{r \to 0} \frac{1}{\gamma(B_r(x))} \int_{B_r(x)} |f(z) - f(x)|e^{-|z|^2} \, dz = 0,$$

for almost every $x$. Let $x$ be a point satisfying (4), then for every $\varepsilon > 0$ there exists $\delta, \ 0 < \delta < 1$ such that

$$\frac{1}{\gamma(B_r(x))} \int_{B_r(x)} |f(z) - f(x)|e^{-|z|^2} \, dz < \varepsilon$$

for $0 < r < \delta$. Set

$$g(z) = \begin{cases} f(z) - f(x), & \text{if } |x - z| \leq \delta, \\ 0, & \text{if } |x - z| > \delta, \end{cases}$$

and notice that $M \gamma g(x) < \varepsilon$. If we set $u_i(y, s) = \gamma_s^{1/2} * f_i(y), \ i = 1, 2$ then we have

$$u(y, s) - f(x) = u_1(y, s) - f_1(x) + u_2(y, s) - f_2(x).$$

We write

$$u_1(y, s) - f_1(x) = \pi^{-n/2} s^{-n/2} \int_{\mathbb{R}^n} \gamma \left( \frac{y - z}{s^{1/2}} \right) (f_1(z) - f_1(x)) \, dz = \pi^{-n/2} s^{-n/2} \left( \int_{|x - z| \leq \delta} + \int_{|x - z| > \delta} \right) = I + II.$$

If $|x| \leq k - 1$ and $(y, s) \in \Gamma(x)$ then by (1) we have

$$|I| \leq M g(x) \leq c_n M \gamma g(x) < c_n \varepsilon.$$

To estimate $II$, observe that if $|x - y| \leq s^{1/2}$ and $\delta \geq 2s^{1/2}$ then

$$|II| \leq \pi^{-n/2} s^{-n/2} \int_{|y - z| \geq \delta^{1/2}} e^{-|y - z|^2/s} |f_1(z)| \, dz + \pi^{-n/2} s^{-n/2} \int_{|x - z| > \delta} e^{-|y - z|^2/s} |f_1(z)| \, dz = II_1 + II_2.$$
We have
\[ II_1 = \pi^{-n/2} s^{-n/2} \int_{\{y-z| \geq \delta/2, \ |z| \leq k\}} \exp \left( -\frac{|y-z|^2}{s} + |z|^2 \right) \left| f(z) \right| e^{-|z|^2} dz \]
\[ \leq \pi^{-n/2} s^{-n/2} \exp((-\delta^2/4s) + k^2) \|f\|_{1,\gamma}, \]
and
\[ II_2 = \|f_1(x)\| \pi^{-n/2} s^{-n/2} \int_{|x-y+t| \geq \delta} e^{-|t|^2/s} dt \]
\[ \leq \|f_1(x)\| \pi^{-n/2} s^{-n/2} \int_{|t| \geq \delta/2} e^{-|t|^2/s} dt \]
\[ = \|f_1(x)\| \pi^{-n/2} \int_{|x| \geq \delta/2s} e^{-|x|^2} dx. \]

If \( |x| \leq k \) then \( f_2(x) = 0 \) and consequently by (1)
\[ |u_2(y, s) - f_2(x)| \leq c_n M_y f_2(x) \]
for \((y, s) \in \Gamma(x)\).

Therefore, collecting estimates and taking supremum over \((y, s) \in \Gamma(x)\), \(0 < s < \alpha\), and letting \( \alpha \to 0 \) we obtain
\[ \Omega f(x) \leq c_n (\varepsilon + M_y f_2(x)), \]
for every \( \varepsilon > 0 \), and a.e. \( x \) with \(|x| \leq k - 1\), which gives (3). Given \( \varepsilon > 0 \), we pick \( k \) such that \( \|f_2\|_{1,\gamma} < c \varepsilon^2 \) then by (3) and the weak-type \((1,1)\) of \( M_y \) we obtain
\[ \gamma \{x : |x| \leq k - 1, \Omega f(x) > \varepsilon\} < \varepsilon, \]
which implies \( \Omega f(x) = 0 \) almost everywhere.

Remarks. (A) Set \( \varphi(x) = (2 \vee |x|)^n \rho^{|x|^2} \) and \( E_\lambda = \{x : \varphi(x) > \lambda\}, \lambda > 0\). We shall show that
\[ \gamma(E_\lambda) \leq c(1 + \ln^+ \lambda)^{n-1}/\lambda. \]
Since \( \gamma \) is a finite measure, it is enough to prove (5) for \( \lambda \) large. The function \( \varphi \) increases with \(|x|\) then \( E_\lambda = \{x : |x| \geq r_0\} \) where \( r_0 \exp(r_0^2) = \lambda \). Therefore \( r_0 \sim (\ln \lambda)^{1/2} \) for \( \lambda \) large. Hence
\[ \gamma(E_\lambda) = c_n \int_{r_0}^{\infty} r^{n-1} e^{-r} dr \]
\[ = c_n e^{-r_0^2} \int_{0}^{\infty} (t + r_0^2)^{(n/2) - 1} e^{-t} dt \]
\[ = c_n e^{-r_0^2} r_0^{n-2} \int_{0}^{\infty} (1 + t/r_0^2)^{(n/2) - 1} e^{-t} dt \]
\[ \sim c_n e^{-r_0^2} r_0^{n-2} = c_n r_0^{2(n-1)}/\lambda, \]
and (5) follows.
(B) The theorem implies that $P_r(x) \to f(x)$ as $r \to 1^-$ a.e. This last result can also be obtained as a consequence of Lemma (2.17) of [C]. The one-dimensional case is due to E. Hille [H].

REFERENCES


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