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A NOTE ON INNER ACTIONS OF HOPF ALGEBRAS

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ABSTRACT. Let H be a commutative, cocommutative, and faithfully projective Hopf algebra over a commutative ring R. A twisted version of inner action of a Hopf algebra, called H-inner action is introduced, and it is shown that H acts H-innerly on an H-Azumaya algebra, if $\mathrm{Pic}(H^*)$ is trivial.

0. INTRODUCTION

Let H be a Hopf algebra over a commutative ring R, and suppose that H acts on an R-algebra A. Sweedler [19] introduced the notion of inner action of H on A: H is said to be acting innerly on A if there exists a convolution invertible $u \in \mathrm{Hom}_R(H, A)$ such that for all $h \in H$, $a \in A$, $h \rightharpoonup a = \Sigma_{(h)} u(h_2) a v(h_{(2)})$, where v is the convolution inverse of u. A natural question is the following: when is a Hopf algebra action inner? The case where R is a field has been studied by several authors, notably [6, 10, 13, 17]; a nice survey was recently published by Montgomery (cf. [16]). In the case of an arbitrary commutative ring, Beattie [4, 5] and Masuoka [15] recently obtained significant results.

Most cases studied in the literature reduce to the case where A is a central simple algebra, or more generally, an Azumaya algebra. Under certain conditions on R, H, and A, one obtains that the action of H on A is inner; in view of the form of the formula stated above, it is not surprising that these results are called "Skolem–Noether-like" theorems. In this note we will show that, at least in the case where H is cocommutative and faithfully projective, this type of result is not only "Skolem–Noether-like", but actually follows from the Skolem-Noether Theorem. We will apply the following trick: if H is cocommutative, then H^* is a commutative R-algebra, and $A \otimes H^*$ is an Azumaya algebra, with groundring H^*, on which we will apply the classical Skolem-Noether Theorem.

Although Proposition 2.7 is a special case of results obtained in [5, 15], we have decided to include full details of our new proof. Actually this method of proof allows us to study Hopf algebra actions on H-dimodule algebras. An

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H-dimodule algebra is an algebra furnished with a compatible H-module and H-comodule structure. For a definition and basic properties of H-modules, H-comodules, and H-dimodules, we refer to the literature (e.g. [1, 14, 20]). We will introduce a kind of "twisted" version of inner action, called H-inner action, and we will show that, if A is an H-Azumaya algebra in the sense of Long [14], and if $\mathrm{Pic}(H^*) = 1$, then H acts H-innerly on A.

1. Notations and preliminary results

Let R be a commutative ring, and H a faithfully projective Hopf algebra, that is, a Hopf algebra which is finitely generated, faithfully flat, and projective as an R-module. The structural maps of H will be denoted by Δ_H (the diagonal), ε_H (the counit), m_H (the multiplication), η_H (the unit), and S_H (the antipode). The subscript H will be omitted whenever no confusion is possible. We will use Sweedler's Σ-notation extensively; for example, we will write, for $h \in H$: $\Delta h = \Sigma_{(h)} h_{(1)} \otimes h_{(2)}$, $(\Delta \otimes 1)\Delta h = (1 \otimes \Delta)\Delta h = \Sigma_{(h)} h_{(1)} \otimes h_{(2)} \otimes h_{(3)}$, etc. For more details, cf. [20].

The group of grouplike elements of H will be denoted by $G(H)$. For the structural map $\psi\colon H \otimes M \to M$ of an H-module M, we will write $\psi(h \otimes m) = h \rightarrow m$, and for the structural map $\chi\colon M \to M \otimes H$ of an H-comodule, we will use Sweedler's notation: $\chi(m) = \Sigma_{(m)} m_{(0)} \otimes m_{(1)}$. Given two H-dimodule algebras A and B, we define the *smash* product $A\#B$ of A and B to be $A \otimes B$ as an H-dimodule, and with algebra structure defined by $(a\#b)(c\#d) = \Sigma_{(b)} a(b_{(1)} \rightarrow c)\#b_{(0)} d$.

The *H-opposite* algebra $\overline{A}$ of an H-dimodule algebra A is equal to A as an H-dimodule, but with multiplication structure given by $\overline{a}.\overline{b} = \overline{\Sigma_{(a)}(a_{(1)} \rightarrow b)a_{(0)}}$.

Suppose that H is a commutative, cocommutative, and faithfully projective Hopf algebra. Long [14] showed that the maps

$$F\colon A\#\overline{A} \to \mathrm{End}_R(A) \quad \text{and} \quad G\colon \overline{A}\#A \to \mathrm{End}_R(A)^{\mathrm{opp}}$$

defined by

$$F(a\#\overline{b})(c) = \Sigma_{(b)} a(b_{(1)} \rightarrow c)b_{(0)} \quad \text{and} \quad G(\overline{a}\#b)(c) = \Sigma_{(c)}(c_{(1)} \rightarrow a)c_{(0)}b$$

are homomorphisms of H-dimodule algebras. If A is faithfully projective as an R-module, and if the maps F and G defined above are isomorphisms, then A is called an H-Azumaya algebra (cf. [2, 14] for more details on H-Azumaya algebras).

The main purpose of this note is to study Hopf algebra actions on H-Azumaya algebras. At first glance, one would expect that an H-Azumaya algebra is nothing else then an Azumaya algebra which is also an H-dimodule algebra; this is not always true, since an H-Azumaya algebra is not necessarily R-central (in fact, it is left and right H-central, and that is a weaker notion). These non-central H-Azumaya algebras are an interesting source of algebras on which a Hopf algebra does not act innerly. Nevertheless, H-Azumaya algebras satisfy properties that are similar to the classical properties of an Azumaya algebra.

For example, they may be classified into a Brauer group that is usually called the Brauer-Long group.

Let $\mathbf{PD}(R, H)$ be the category of invertible H-dimodules and H-dimodule homomorphisms; then the Grothendieck group of this category may be viewed as a dimodule version of the Picard group:

$$PD(R, H) = K_0\mathbf{PD}(R, H).$$

This invariant of R and H may be easily linked to $\mathrm{Pic}(R)$, H and H^*:

1.1. **Proposition** [7, Proposition 1.6]. $PD(R, H) \cong \mathrm{Pic}(R) \times G(H) \times G(H^*)$.

Proof. For details, we refer to [7, Proposition 1.6]. We restrict this paper to giving a description of the isomorphism α: $\mathrm{Pic}(R) \times G(H) \times G(G^*) \to PD(R, H)$. For $([I], h, h^*) \in \mathrm{Pic}(R) \times G(H) \times G(H^*)$, we define $\alpha([I], h, h^*) = [I(h, h^*)]$, where $I(h, h^*)$ is equal to I as an R-module, and with dimodule structure defined by $\chi(x) = x \otimes h$ for all $x \in I$, and $k \rightharpoonup x = h^*(k)x$ for all $k \in H$, $x \in I$.

For an Azumaya algebra over a ring with trivial Picard group, we have the Skolem–Noether theorem, that is, every automorphism of A is inner. For an H-Azumaya algebra, this is not always true, and this is why we introduce the notions of H-inner and H-INNER automorphisms. As before, let H be a faithfully projective, commutative, and cocommutative Hopf algebra, and let A be an H-Azumaya algebra. Let H-$\mathrm{Aut}(A)$ be the group of all H-dimodule R-algebra automorphisms of A. We call $f \in H$-$\mathrm{Aut}(A)$ H-INNER, if there exists an invertible $x \in A$ such that

(1) $f(a) = xax^{-1}$, for all $a \in A$;
(2) $\chi(x) = x \otimes 1$;
(3) $h \rightharpoonup x = \varepsilon(h)x$, for all $h \in H$.

The subgroup of H-$\mathrm{Aut}(A)$ consisting of H-INNER automorphisms of A will be denoted by H-$\mathrm{INN}(A)$. $f \in H$-$\mathrm{Aut}(A)$ will be called H-inner if there exists an invertible $x \in A$, and $h^* \in G(H^*)$ such that for all $a \in A$: $f(a) = \Sigma_{(a)}h^*(a_{(1)})xa_{(0)}x^{-1}$. We have the following generalization of the Rosenberg–Zelinsky exact sequence:

1.2. **Proposition** (Dimodule version of the Rosenberg–Zelinsky sequence). *For an H-Azumaya algebra A, we have the following short exact sequence*:

$$1 \to H\text{-}\mathrm{INN}(A) \to H\text{-}\mathrm{Aut}(A) \xrightarrow{\Phi} PD(R, H)$$

where $\Phi(f) = I_f$, *with*

$$I_f = \{x \in A: \Sigma_{(a)}(a_{(1)} \rightharpoonup x)a_{(0)} = f(a)x,\ \text{for all } a \in A\}.$$

Proof. We leave it as an exercise to the reader to modify the proof of the exactness of the classical Rosenberg-Zelinsky sequence, as presented for example in [12, IV.1.2]. Let us remark that instead of using the classical Morita equivalence, one has to use the H-dimodule version of the Morita equivalences as it is discussed by Beattie in [2, 3.6].

1.3. **Corollary** (Generalized Skolem–Noether Theorem). *If* $\mathrm{Pic}(R) = 1$, *then every* $f \in H\text{-}\mathrm{Aut}(A)$ *in* H*-inner.*

Proof. Since $[I_f] = 1$ in $\mathrm{Pic}(R)$, $I_f = Rx$ for some invertible $x \in A$. Let $[I_f] = ([R], h, h^*)$ in $PD(R, H) \cong \mathrm{Pic}(R) \times G(H) \times G(H^*)$ (cf. Proposition 1.3). Then $k \rightharpoonup x = h^*(k)x$ for all $k \in H$, and the result follows from the preceding theorem.

2. Hopf algebra actions on H-Azumaya algebras

2.1. **The Hopf algebra** $\mathrm{Hom}_R(H, K)$. Let H, K be faithfully projective Hopf algebras, and suppose that H is cocommutative. $\underline{H}^*$ will be H^*, viewed as a commutative R-algebra (that is, we forget the coalgebra structure of H^*).

Let $\mathscr{H} = \mathrm{Hom}_R(H, K)$. An $\underline{H}^*$-Hopf algebra structure may be defined on $\mathscr{H}$ as follows: the multiplication is given by the convolution, and the unit map $\eta_{\mathscr{H}}: \underline{H}^* \to \mathscr{H}$ is given by $\eta_{\mathscr{H}}(h^*) = \eta_K \circ h^*$. The diagonal map

$$\Delta_{\mathscr{H}}: \mathscr{H} \to \mathscr{H} \otimes_{\underline{H}^*} \mathscr{H} \cong \mathrm{Hom}_R(H, K \otimes K)$$

is defined by

$$\Delta_{\mathscr{H}}(\mu) = \Delta_K \circ \mu$$

for all $\mu \in \mathscr{H}$. The counit $\varepsilon_{\mathscr{H}}: \mathscr{H} \to \underline{H}^*$ is given by $\varepsilon_{\mathscr{H}}(\mu) = \varepsilon_K \circ \mu$, and the antipode $S_{\mathscr{H}}: \mathscr{H} \to \mathscr{H}$ by $S_{\mathscr{H}}(\mu) = S_K \circ \mu$.

2.2. **Lemma.** *We have an isomorphism of* $\underline{H}^*$*-Hopf algebras* $\alpha: \underline{H}^* \otimes K \to \mathscr{H}$ *given by*

$$\alpha(h^* \otimes k)(h) = h^*(h)k,$$

for all $h^* \in H^*$, $k \in K$, *and* $h \in H$.

Proof. Using the hom-tensor relations, we obtain that $\underline{H}^* \otimes K = \mathrm{Hom}_R(H, R) \otimes K \cong \mathrm{Hom}_R(H, K) = \mathscr{H}$ as R-modules, and the isomorphism is given by α. It is straightforward to check that α preserves the structure maps.

2.3. **Lemma.** $G(\mathscr{H}) = \{\mu \in \mathscr{H} : \mu$ *is a coalgebra homomorphism*$\}$.

Proof. μ is grouplike if and only if $\Delta_{\mathscr{H}}(\mu) = \mu \otimes \mu$ and $\varepsilon_{\mathscr{H}}(\mu) = \varepsilon_H$, the unit in $\underline{H}^*$. The latter condition is equivalent to $\varepsilon_K \circ \mu = \varepsilon_H$. We have seen above that

$$(\Delta_{\mathscr{H}}(\mu))(h) = \Delta_K(\mu(h)) = \Sigma_{\mu(h)} \mu(h)_{(0)} \otimes \mu(h)_{(1)}$$

$\mu \to \mu \otimes \mu$ may be viewed as a map

$$\begin{aligned} \mathscr{H} = \mathrm{Hom}_R(H, K) &\cong \underline{H}^* \otimes K \to (\underline{H}^* \otimes K) \otimes_{\underline{H}^*} (\underline{H}^* \otimes K) \\ &\cong \underline{H}^* \otimes K \otimes K \cong \mathrm{Hom}_R(H, K \otimes K). \end{aligned}$$

One may easily establish that $\mu \otimes \mu$ is given by $(\mu \otimes \mu)(h) = \Sigma_{(h)} \mu(h_{(0)}) \otimes \mu(h_{(1)})$, for all $h \in H$ and the result follows.

2.4. **Lemma.** *$\mathscr{H}^{\clubsuit} = \mathrm{Hom}_R(H, K)^{\clubsuit} \cong \mathrm{Hom}_R(H, K^*)$, where $\clubsuit$ means the dual with respect to the groundring $\underline{H}^*$.*

Proof. The result follows quite easily:

$$\mathscr{H}^{\clubsuit} = \mathrm{Hom}_R(H, K)^{\clubsuit} \cong (\underline{H}^* \otimes K)^{\clubsuit} \cong \underline{H}^* \otimes K^* \cong \mathrm{Hom}_R(H, K^*).$$

For later use, we give an explicit description of the action of $\mathscr{H}^{\clubsuit}$ on $\mathscr{H}$. The action of $\underline{H}^* \otimes K^*$ on $\underline{H}^* \otimes K$ is given by

$$\langle h^* \otimes k^*, g^* \otimes k\rangle = k^*(k)h^* * g^*,$$

for all $h^*, g^* \in H^*$, $k \in K$, $k^* \in K^*$. Translating this into an action of $\mathscr{H}^{\clubsuit}$ on $\mathscr{H}$, we obtain, for $\mu^* \in \mathscr{H}^{\clubsuit}$, $\mu \in \mathscr{H}$: $\langle \mu^*, \mu\rangle \in \underline{H}^*$ is defined by

$$\langle \mu^*, \mu\rangle(h) = \Sigma_{(h)}\langle \mu^*(h_{(1)}), \mu(h_{(2)})\rangle$$

for all $h \in H$.

Now, suppose that A is a K-module algebra; then $\mathscr{A} = \mathrm{Hom}_R(H, A)$ is an $\mathscr{H}$-module algebra. The algebra structure on $\mathscr{A}$ is given by the convolution, and the $\mathscr{H}$-action is given by

$$(\mu \rightharpoonup u)(h) = \Sigma_{(h)}\mu(h_{(1)}) \rightharpoonup u(h_{(2)}))$$

for all $h \in H$, $\mu \in \mathscr{H}$, $u \in \mathscr{A}$.

The proof of the following lemma is similar to the proof of Lemma 2.2.

2.5. **Lemma.** *$\underline{H}^* \otimes A \cong \mathscr{A}$ as $\mathscr{H}$-module algebras.*

Next, suppose that $H = K$, and define $F, G: \mathscr{A} \to \mathscr{A}$ by

$$F(f)(h) = \Sigma_{(h)}(h_{(1)} \rightharpoonup f(h_{(2)})) \quad \text{and} \quad G(f)(h) = \Sigma_{(h)}(\mathrm{S}(h_{(1)}) \rightharpoonup f(h_{(2)})),$$

for all $f \in \mathscr{A}$, $h \in H$.

2.6. **Lemma.** *$F, G \in \mathscr{H}$-$\mathrm{Aut}(\mathscr{A})$, and F and G are each other's algebra automorphism inverses.*

Proof. First, let us show that F preserves the convolution. For all $f, g \in \mathscr{A}$ and $h \in H$ we have that

$$\begin{aligned} F(f * g)(h) &= \Sigma_{(h)}h_{(1)} \rightharpoonup (f(h_{(2)})g(h_{(3)})) \\ &= \Sigma_{(h)}(h_{(1)} \rightharpoonup f(h_{(2)}))(h_{(3)} \rightharpoonup g(h_{(4)})) \\ &= (F(f) * F(g))(h) \end{aligned}$$

If $h^* \in \underline{H}^*$, then we have for all $h \in H$:

$$\begin{aligned} F(h^*)(h) &= \Sigma_{(h)}h_{(1)} \rightharpoonup h^*(h_{(2)}) \\ &= \Sigma_{(h)}\varepsilon(h_{(1)})h^*(h_{(2)}) \\ &= h^*(h), \end{aligned}$$

so $F(h^*) = h^*$.

Finally, it is straightforward to show that $G \circ F = F \circ G = \mathrm{Id}$.

We have an embedding $i: A \to \underline{H}^* \otimes A \cong \mathscr{A}$, given by $i(a)(h) = \varepsilon(h)a$ for all $a \in A$, $h \in H$. Similarly, we have an embedding $i: H \to \mathscr{H}$.

Also observe that $F(i(a))(h) = \Sigma_{(h)} h_{(1)} \rightharpoonup \varepsilon(h_{(2)})a = h \rightharpoonup a$ for all $h \in H$, $a \in A$.

As an application, let us give an elementary proof of the following (cf. [5, 15] for a different approach):

2.7. **Proposition.** *Let H be a cocommutative faithfully projective Hopf algebra, and suppose that $\mathrm{Pic}(\underline{H}^*) = 1$. Then H acts innerly on any H-module Azumaya algebra A.*

Proof. Take $H = K$ in the above arguments. $\underline{H}^* \otimes A = \mathscr{A}$ is an $\underline{H}^*$-module Azumaya algebra, and therefore, the automorphism F defined above is inner, by the (classical!) Skolem–Noether Theorem. So there exists $u \in \mathrm{Hom}_R(H, A)$, with convolution inverse v, say, such that $F(f) = u * f * v$, or $F(f)(h) = \Sigma_{(h)} u(h_{(1)}) f(h_{(2)}) v(h_{(3)})$ for all $f \in \mathscr{A}$, $h \in H$. Applying this formula to $f = i(a)$, we obtain:

$$h \rightharpoonup a = F(i(a))(h) = \Sigma_{(h)} u(h_{(1)}) \varepsilon(h_{(2)}) a v(h_{(3)}) = \Sigma_{(h)} u(h_{(1)}) a v(h_{(2)}),$$

so H acts innerly on A.

The argument used in the proof of Proposition 2.7 may be generalized in order to study the action of a Hopf algebra H on an H-Azumaya algebra. From now on, suppose that $H = K$ is a commutative, cocommutative faithfully projective Hopf algebra. We already know that $\mathscr{A} = \mathrm{Hom}_R(H, A)$ is an $\mathscr{H}$-module algebra. Now $\mathscr{A}$ has the structure of an $\mathscr{H}$-dimodule algebra if we define $\chi_{\mathscr{A}}: \mathscr{A} \to \mathscr{A} \otimes_{\mathscr{H}^*} \mathscr{H}$ by $\chi_{\mathscr{A}}(u) = \chi_A \circ u$. The isomorphism discussed in Lemma 2.5 is then an isomorphism of $\mathscr{H}$-dimodule algebras, and $\mathscr{A}$ is an $\mathscr{H}$-Azumaya algebra.

2.8. **Theorem.** *Let A be an H-Azumaya algebra. If $\mathrm{Pic}(H^*) = 1$, then H acts H-innerly on A, that is, there exists a convolution invertible $u \in \mathrm{Hom}_R(H, A)$, with convolution inverse v, and $\mu^* \in \mathrm{Hom}_{\mathrm{bialg}}(H, H^*)$ such that for all $h \in H$, $a \in A$:*

$$h \rightharpoonup a = \Sigma_{(h),(a)} \mu^*(h_{(1)})(a_{(1)}) u(h_{(2)}) a_{(0)} v(h_{(3)}).$$

Proof. With notations as above, $\mathscr{A}$ is an $\mathscr{H}$-Azumaya algebra. We apply Proposition 1.2 to the automorphism F. Then $I_F \cong \underline{H}^*(\mu, \mu^*)$ with $\mu \in G(\mathscr{H})$, $\mu^* \in G(\mathscr{H}^{\clubsuit})$. Let I_F be generated $u \in \mathscr{A} = \mathrm{Hom}_R(H, A)$. Then u is convolution invertible, and, for all $\lambda \in \mathscr{H}$, $\lambda \rightharpoonup u = \langle \mu^*, \lambda \rangle * u$ (cf. Proposition 1.1). For all $f \in \mathscr{A}$ we have that

$$F(f) * u = \Sigma_{(f)} (f_{(1)} \rightharpoonup u) * f_{(0)},$$

and therefore,

$$F(f) = \Sigma_{(f)} (f_{(1)} \rightharpoonup u) * f_{(0)} * v.$$

Let $f = i(a)$ with $a \in A$. Then for all $h \in H$,

$$\chi_{\mathscr{A}}(i(a))(h) = \chi_A(\varepsilon(h)(a)) = \Sigma_{(a)}\varepsilon(h)a_{(0)} \otimes a_{(1)},$$

so

$$\chi_{\mathscr{A}}(i(a)) = \Sigma_{(a)} i(a_{(0)}) \otimes i(a_{(1)}) \in \mathscr{A} \otimes_{\underline{H}^*} \mathscr{H}.$$

Now $i(a) \to u = \langle \mu^*, i(a)\rangle * u$, and for all $h \in H$,

$$\begin{aligned}\langle \mu^*, i(a)\rangle(h) &= \Sigma_{(h)}\mu^*(h_{(1)})(i(a)(h_{(2)}))\\ &= \Sigma_{(h)}\mu^*(h_{(1)})(\varepsilon(h_{(2)})a)\\ &= \mu^*(h)(a).\end{aligned}$$

Therefore,

$$F(i(a)) = \Sigma_{(a)}\mu^*(\cdot)(a_{(1)}) * u * i(a_{(0)}) * v,$$

and for all $h \in H$,

$$F(i(a))(h) = h \to a = \Sigma_{(a),(h)}\mu^*(h_{(1)})(a_{(1)})u(h_{(2)})a_{(0)}v(h_{(3)}).$$

It follows from Lemma 2.3 that μ^* is a coalgebra homomorphism, so the only thing that remains to be shown is that μ^* is an algebra homomorphism.

Let ρ be the composition of the maps

$$\begin{aligned}\rho\colon \operatorname{Hom}_R(H\otimes H, H) &- \operatorname{Aut}(\operatorname{Hom}_R(H\otimes H, A))\\ &\xrightarrow{\Phi} PD((\underline{H\otimes H})^*, \operatorname{Hom}_R(H\otimes H, H))\\ &\to G(\operatorname{Hom}_R(H\otimes H, H^*)).\end{aligned}$$

Consider the following automorphisms F_0, F_1, F_2 of $\operatorname{Hom}_R(H\otimes H, A)$:

$$\begin{aligned}F_0(f)(h\otimes k) &= \Sigma_{(k)}k_{(1)} \to f(h\otimes k_{(2)}),\\ F_1(f)(h\otimes k) &= \Sigma_{(h),(k)}h_{(1)}k_{(1)} \to f(h_{(2)}\otimes k_{(2)}),\\ F_2(f)(h\otimes k) &= \Sigma_{(h)}h_{(1)} \to f(h_{(2)}\otimes k).\end{aligned}$$

If $u \in I_F$, then $\varepsilon\otimes u \in I_{F_0}$, $u\circ m_H \in I_{F_1}$, $u\otimes\varepsilon \in I_{F_2}$, and therefore,

$$\rho(F_0) = \varepsilon\otimes\mu^*; \qquad \rho(F_1) = \mu^*\circ m_H; \qquad \rho(F_2) = \mu^*\otimes\varepsilon.$$

Now $F_2F_1^{-1}F_0$ is the identity, so

$$\rho(F_2)\rho(F_1)^{-1}\rho(F_0) = \varepsilon.$$

It follows that for all $h, k \in H$,

$$(\mu^*\circ m_H)(h\otimes k) = ((\varepsilon\otimes\mu^*) * (\mu^*\otimes\varepsilon))(h\otimes k),$$

and

$$\mu^*(hk) = \mu^*(h) * \mu^*(k)$$

2.9. **Examples.**

(2.9.1) Let G be a finite abelian group, and let $H = RG$. Then

$$\operatorname{Hom}_{\text{bialg}}(H, H^*) \cong \operatorname{Hom}(G(H), G(H^*)) = \operatorname{Hom}(G, G^*),$$

and for an RG-Azumaya algebra A,

$$\mathrm{Hom}_R(H, A) \cong \mathrm{Map}(G, A)$$
$$\mathbb{G}_m(\mathrm{Hom}_R(H, A)) \cong \mathrm{Map}(G, \mathbb{G}_m(A)).$$

It is also clear that the convoiution on $\mathrm{Hom}_R(H, A)$ corresponds to pointwise multiplication on $\mathrm{Map}(G, A)$. Suppose that $\mathrm{Pic}(R) = 1$; then $\mathrm{Pic}((RG)^*) = \mathrm{Pic}(GR)$ is the direct sum of $|G|$ copies of $\mathrm{Pic}(R)$, so $\mathrm{Pic}((GR)^*)$ is trivial. Theorem 2.8 implies that there exists $\mu^* \in \mathrm{Hom}(G, G^*)$ and for all $\sigma \in G$, $u_\sigma \in \mathbb{G}_m(A)$ such that for all homogeneous $a \in A$: $\sigma \cdot a = \mu^*(\sigma)(\alpha)u_\sigma a u_\sigma^{-1}$, where α is the grade of a; this result has been exploited by the author and M. Beattie in [8].

(2.9.2) Let us give a more specific example: let $R = \mathbb{Z}[\sqrt{2}]$ and $H = R[x]/(x^2 - \sqrt{2}x)$, with coalgebra structure defined by $\Delta x = x \otimes 1 + 1 \otimes x - \sqrt{2}x \otimes x$, $\varepsilon x = 0$, and $S = \mathrm{Id}$. For a more detailed discussion of this Hopf algebra, we refer to [9, 11].

The map $f: H \to H^*$, defined by $f(1) = \varepsilon$ and $f(x) = y$, where $y(x) = -1$, is an isomorphism of Hopf algebras. Let A be equal to H as an R-algebra, and define an H-module and H-comodule structure on A by $\chi(h) = \Delta h$, for all $h \in H$, $x \rightharpoonup 1 = 0$, and $x \rightharpoonup x = \sqrt{2}x - 1$. Then A is an H-Azumaya algebra (cf. [7, Proposition 4.3]); a lengthy, but straightforward computation shows that the formula Theorem 2.8 holds if we take $u = v = \mathrm{Id}$ and $\mu^*: H \to H^*$ equal to the map f defined above.

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