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ON THE WEIGHTED ESTIMATE OF THE SOLUTION ASSOCIATED WITH THE SCHRÖDINGER EQUATION

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ABSTRACT. Let $u(x, t)$ be the solution of the Schrödinger equation with initial data f in the Sobolev space $H^{-1+a/2}(\mathbb{R}^n)$ with $a > 1$. This paper shows that the weighted inequality $\int_{\mathbb{R}^n}\int_{\mathbb{R}}|u(x, t)|^2\,dt(1+|x|)^{-a}\,dx \leq C\|f\|_{H^{-1+a/2}(\mathbb{R}^n)}$ is false. Another improved weighted inequality is proved for the general case.

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Let f belong to the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ and set

$$u(x, t) = \int_{\mathbb{R}^n} e^{ix\cdot\varepsilon}e^{it|\varepsilon|^2}\hat{f}(\varepsilon)\,d\varepsilon, \qquad x \in \mathbb{R}^n, \ t \in \mathbb{R}. \tag{1.1}$$

Here $\hat{f}$ denotes the Fourier transform of f, defined by

$$\hat{f}(\varepsilon) = \int_{\mathbb{R}^n} e^{-ix\cdot\varepsilon} f^{(x)}\,dx.$$

It is well known that $u(x, t)$ is the solution of the Schrödinger equation with the initial data f:

$$\Delta u = i\partial u/\partial t, \qquad t > 0, \ u(x, 0) = f(x).$$

For $s \in \mathbb{R}$ we also introduce Sobolev spaces $H^s(\mathbb{R}^n)$ by setting

$$H^s(\mathbb{R}^n) = \left\{ f + \mathscr{S}'(\mathbb{R}^n) \colon \|f\|_{H^s(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n}(1+|x|^2)^s|\hat{f}(x)|^2\,dx\right)^{1/2} < \infty \right\}.$$

In [V] the following result of maximal operator $u^*(x) = sup_{|t|>0}|u(x, t)|$ was established for functions in the Sobolev space $H^{2\alpha-1+a/2}(\mathbb{R}^n)$.

Theorem A [V, Theorem 2]. *Let f be in $H^s(\mathbb{R}^n)$ with $s > a/2$ and $a > 1$. Then*

$$\left(\int |u^*(x)|^2 \frac{dx}{(1+|x|)^a}\right)^{1/2} \leq C\|f\|_{H^s(\mathbb{R}^n)}. \tag{1.2}$$

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In a crucial way, the proof of this theorem uses the following classical Sobolev inequalities which states that the $H^{\gamma}(\mathbb{R})$ with $\gamma > 1/2$ is embedded in $L^{\infty}(\mathbb{R})$ and the following:

Theorem B [V, Theorem 3]. *If* $\alpha \geq 0$ *and* $a > 1$, *then*

$$\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}}\left|\frac{\partial^{\alpha}}{\partial t^{\alpha}}u(x,t)\right|^2 dt\frac{dx}{(1+|x|)^a}\right)^{1/2} \leq C\|f\|_{H^{2\alpha-1+a/2}(\mathbb{R}^n)}. \tag{1.3}$$

But the proof of Theorem B is slightly in error with $\alpha = 0$, thus placing the validity of Theorem B in doubt when $\alpha = 0$. The purpose of this note is to show by counterexample, that estimates (1.3) cannot be expected to hold true for $\alpha = 0$.

Theorem 1. *The inequality in Theorem* B *with* $\alpha = 0$, *i.e.*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}}|u(x,t)|^2 dt\frac{dx}{(1+|x|)^a} \leq C\|f\|^2_{H^{-1+a/2}(\mathbb{R}^n)} \tag{1.4}$$

does not hold for some $f \in H^{-1+a/2}(\mathbb{R}^n)$. *In fact, for* $n \geq 2$ *there exists an* $f_0 \in H^{-1+a/2}(\mathbb{R}^n)$ *so that*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}}|u(x,t)|^2 dt\frac{dx}{(1+|x|)^a} = \infty. \tag{1.5}$$

For $n = 1$, *the corresponding inequality*

$$\left\|\left(\int_{-\infty}^{\infty}|u(x,t)|^2 dt\right)^{1/2}\right\|_{L^{\infty}(\mathbb{R})} \leq C\|f\|_{H^{-1}(\mathbb{R})} \tag{1.6}$$

is also false with some $f \in H^{-1}(\mathbb{R})$. *Indeed, there is also an* $f_0 \in H^{-1}(\mathbb{R})$ *so that* (1.6) *fails to be true.*

It may be interesting to determine the source of the error in Theorem B. The proof of Theorem B makes use of the following lemma from [V]:

Lemma. *Let* g *be in* $L^2(S^{n-1})$. *Then if* $a > 1$,

$$\left(\int_{\mathbb{R}^n}\left|\int_{S^{n-1}}g(\varepsilon)e^{ix\cdot\varepsilon}\,d\sigma(\varepsilon)\right|^2\frac{dx}{(1+|x|)^a}\right)^{1/2} \leq C\left(\int_{S^{n-1}}|g(\varepsilon)|^2\,d\sigma(\varepsilon)\right)^{1/2}, \tag{1.7}$$

S^{n-1} *being the unit sphere in* $\mathbb{R}^n$ *and* $d\sigma(\varepsilon)$ *the Lebesgue measure on* S^{n-1}.

This lemma was proved in [V] only for $n = 2$ which is heavily dependent on geometry. The second purpose of this note is to give a proof in the general case. In fact, the following stronger result can be established.

Theorem 2. *Let* g *be in* $L^2(S^{n-1})$. *Then if* $a > 1$, *we have*

$$\left(\int_{\mathbb{R}^n}\left|\int_{S^{n-1}}g(\varepsilon)e^{ix\cdot\varepsilon}\,d\sigma(\varepsilon)\right|^2\frac{dx}{|x|^a}\right)^{1/2} \leq C_a\left(\int_{S^{n-1}}|g(\varepsilon)|^2\,d\sigma(\varepsilon)\right)^{1/2}. \tag{1.8}$$

Theorem 2 leads to the conclusion that the estimate from line 24 to line 25 is wrong for $\alpha = 0$ in the proof of Theorem B. (See [V, p. 875].) Thus, it would not seem proper to prove Theorem A by using (1.3) for $\alpha = 0$.

We shall first give a proof of Theorem 1 in §2. The proof of Theorem 2 is postponed to §3. The constants C need not be the same at each occurrence.

2. Proof of Theorem 1

First suppose that $n \geq 2$. Let $f_0(x) = f_0(|x|)$ be a radial function that belongs to $L^2(\mathbb{R}^2)$ and its Fourier transform

$$\hat{f}_0(|x|) = |x|^{-\sigma - n/2 + 1}(1 + |x|)^{-\beta},$$

where

$$a/2 < \sigma < 1 \tag{2.1}$$

and

$$n < \beta. \tag{2.2}$$

Then it is not difficult to verify that $f_0 \in L^2(\mathbb{R}^n)$ and $f_0 \in H^{-1+a/2}(\mathbb{R}^n)$, since

$$\int_{\mathbb{R}^n} |f_0(x)|^2\, dx = \int_{\mathbb{R}^n} |\hat{f}_0(x)|^2\, dx = \int_{\mathbb{R}^n} |x|^{-2\sigma - n + 2}(1 + |x|)^{-2\beta}\, dx < +\infty$$

and

$$\begin{aligned}\int_{\mathbb{R}^n} &(1 + |x|^2)^{-1+a/2} |\hat{f}_0(x)|^2\, dx \\ &= \int_{\mathbb{R}^n} (1 + |x|^2)^{-1+a/2} |x|^{-2\sigma - n + 2}(1 + |x|)^{-2\beta}\, dx < +\infty\end{aligned}$$

given conditions (2.1) and (2.2).

On the other hand, with a simple change of variable, by (1.1) we get the following representation of $u(x, t)$ in polar coordinates

$$u(x, t) = \frac{1}{2}\int_0^\infty e^{is \cdot t} s^{(n-2)/2} \int_{S^{n-1}} \hat{f}_0(s^{1/2}\varepsilon) e^{is^{1/2} x \cdot \varepsilon}\, d\sigma(\varepsilon)\, ds.$$

Using Plancherel's theorem in the t variable, it follows that

$$\begin{aligned}\int_{\mathbb{R}^n} \int_{\mathbb{R}} &|u(x, t)|^2\, dt \frac{dx}{(1 + |x|)^a} \\ &= \frac{1}{4} \int_{\mathbb{R}^n} \int_0^\infty \left| s^{(n-2)/2} \int_{S^{n-1}} \hat{f}_0(s^{1/2}\varepsilon) e^{is^{1/2} x \cdot \varepsilon}\, d\sigma(\varepsilon) \right|^2 ds \frac{dx}{(1 + |x|)^a} \\ &= \frac{1}{4} \int_{\mathbb{R}^n} \int_0^\infty \left| s^{(n-2)/2} \hat{f}_0(s^{1/2}) \int_{S^{n-1}} e^{is^{1/2} x \cdot \varepsilon}\, d\sigma(\varepsilon) \right|^2 ds \frac{dx}{(1 + |x|)^a} \\ &= \frac{(2\pi)^n}{4} \int_{\mathbb{R}^n} \int_0^\infty s^{n-2} |\hat{f}_0(s^{1/2})|^2 \frac{\left|J_{(n-2)/2}(s^{1/2}|x|)\right|^2}{|x|^{n-2}}\, ds \frac{dx}{(1 + |x|)^a}.\end{aligned} \tag{2.3}$$

In the last equality we used the fact that [SW, p. 154]

$$\int_{S^{n-1}} e^{ix\cdot\varepsilon}\, d\sigma(\varepsilon) = (2\pi)^{n/2} J_{(n-2)/2}(|x|)/|x|^{(n-2)/2},$$

where $J_k(x)$ is Bessel's function of order k. Using the expression for $\hat{f}_0$ we get from (2.3) that

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}} |u(x,t)|^2\, dt \frac{dx}{(1+|x|)^a} \\
&= c_n \int_0^\infty s^{n/2-2} |\hat{f}_0(s^{1/2})|^2 \int_0^\infty |J_{(n-2)/2}(\gamma)|^2 \frac{\gamma\, d\gamma}{(1+\gamma/\sqrt{s})^a}\, ds \\
&\geq c_n \int_0^1 s^{n/2-2+a/2} |\hat{f}_0(s^{1/2})|^2 \int_1^\infty |J_{(n-2)/2}(\gamma)|^2 \gamma^{-(a-1)}\, d\gamma\, ds \\
&= c_n \int_1^\infty |J_{(n-2)/2}(\gamma)|^2 \gamma^{-(a-1)}\, d\gamma \cdot \int_0^1 s^{n/2-2+a/2} |\hat{f}_0(s^{1/2})|^2,\, ds = +\infty,
\end{aligned}$$

since the last integral

$$\int_0^1 s^{n/2-2+a/2} |\hat{f}_0(s^{1/2})|^2\, ds = \int_0^1 s^{-1+a/2-\sigma}(1+s^{1/2})^{-2\beta}\, ds = +\infty$$

given condition (2.1). Thus the proof of the first part of Theorem 1 is complete.

For $n = 1$ the corresponding inequality (1.6) is false with the following function $f_0 \in H^1(\mathbb{R})$:

$$\hat{f}_0(x) = |x|^{-\sigma}(1+|x|)^{-\beta} \qquad (0 < \sigma < 1/2,\ \sigma + \beta > 1/2)$$

can also be easily verified. We omit the detail.

3. Proof of Theorem 2

Let us develop $g(\varepsilon) \in L^2(S^{n-1})$ into a series of spherical harmonics

$$g(\varepsilon) \sim \sum_{k=0}^{\infty} a_k Y_k(\varepsilon) \qquad (\varepsilon \in S^{n-1}), \tag{3.1}$$

where $Y_k(\varepsilon)$ is a spherical function of order k, i.e., the value on S^{n-1} of a homogeneous polynomial $P(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ satisfying Laplace's equation $\Delta P = 0$. We may always normalize the $Y_k(\varepsilon)$ and assume that

$$\|Y_k\| = \left(\int_{S^{-1}} |Y_k(\varepsilon)|^2\, d\sigma(\varepsilon)/|S^{n-1}|\right)^{1/2} = 1,$$

$|S^{n-1}|$ being the Lebesgue measure of S^{n-1}. Thus the functions $Y_k(\varepsilon)$ form an orthonormal system on S^{n-1} and Bessel's inequality gives

$$\left(\sum_{k=0}^{\infty} |a_k|^2\right)^{1/2} \leq \left(\int_{S^{n-1}} |g(\varepsilon)|^2\, d\sigma(\varepsilon)\right)^{1/2}. \tag{3.2}$$

Our first step will be to replace the function $g(\varepsilon)$ in (1.8) by the development (3.1) and prove the equation

$$\int_{S^{n-1}} g(\varepsilon)e^{ix\cdot\varepsilon}\,d\sigma(\varepsilon) = \sum_{k=0}^{\infty} a_k \int_{S^{n-1}} Y_k(\varepsilon)\cdot e^{ix\cdot\varepsilon}\,d\sigma(\varepsilon). \tag{3.3}$$

Since $g \in L^2(S^{n-1})$ and the development (3.1) converges to g over S^{n-1}, (3.3) follows in norm L^2 by Schwarz' inequality;

Now we invoke the formulas [AH, p. 572]

$$\int_{S^{n-1}} Y_k(\varepsilon)e^{ix\cdot\varepsilon}\,d\sigma(\varepsilon) = (2\pi)^{\lambda+1} i^k Y_k(x')\cdot J_{k+\lambda}(|x|)/|x|^{\lambda} \qquad (\lambda = (n-2)/2)$$

and (3.3), so we get the equality as follows

$$\left|\int_{S^{n-1}} g(\varepsilon)e^{ix\cdot\varepsilon}\,d\sigma(\varepsilon)\right|^2 = c_k \sum_{k,l=0}^{\infty} (-1)^l i^{k+l} a_k\overline{a}_l Y_k(x')\overline{Y}_l(x')\cdot J_{k+\lambda}(|x|)J_{l+\lambda}(|x|)/|x|^{2\lambda},$$

where $x' = x/|x| \in S^{n-1}$. Thus, the left-hand side of (1.8) is equal to

$$c_n\int_0^{\infty}\int_{S^{n-1}} \sum_{k,l=0} (-1)^l i^{k+l} a_k\overline{a}_l Y_k(x')\overline{Y}_l(x')\,d\sigma(x') J_{k+\lambda}(\gamma)\cdot \overline{J}_{l+\lambda}(\gamma)\cdot\gamma^{1-a}\,d\gamma$$
$$= c_n\sum_{k=0}^{\infty}|a_k|^2\int_0^{\infty}|J_{k+\lambda}(\gamma)|^2\gamma^{1-a}\,d\gamma.$$

The last integral was evaluated in [L, 13.4.2(3)]:

$$\int_0^{\infty} t^{-d}J_{\mu}(bt)J_{\nu}(bt)\,dt = \frac{(b/2)^{d-1}\Gamma(\lambda)\Gamma\left(\frac{\mu+\nu-d+1}{2}\right)}{2\Gamma\left(\frac{\nu-\mu+d+1}{2}\right)\Gamma\left(\frac{\nu+\mu+d+1}{2}\right)\Gamma\left(\frac{\mu-\nu+d+1}{2}\right)}$$

under the conditions

$$\operatorname{Re}(\mu+\nu+1) > \operatorname{Re}(d) > 0, \qquad b > 0.$$

In our case the last integral in (3.4) is

$$\int_0^{\infty}|J_{k+\lambda}(\gamma)|^2\gamma^{1-a}\,d\gamma = \frac{1}{2^{a-1}}\frac{\Gamma(a-1)}{\Gamma^2(a/2)}\frac{\Gamma(k+\frac{n}{2}-\frac{a}{2})}{\Gamma(k+\frac{n}{2}-1+\frac{a}{2})}$$
$$\le c/(k+(n-a)/2)^{a-1} \le c/k^{a-1} \qquad (k = 1, 2, \ldots).$$

Then Theorem 2 follows from (3.2) to (3.5).

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