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WEAK COMPACTNESS IN $L^1(\mu, X)$

A. ÜLGER

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ABSTRACT. Let (Ω, Σ, μ) be a probability space, X a Banach space, and $L^1(\mu, X)$ the Banach space of Bochner integrable functions $f: \Omega \to X$. Let $W = \{f \in L^1(\mu, X): \text{ for a.e. } \omega \text{ in } \Omega,\ \|f(\omega)\| \le 1\}$. In this paper we characterize the rwc (relatively weakly compact) subsets of $L^1(\mu, X)$. The main results are as follows:

Theorem A. *A subset H of W is rwc iff given any sequence (f_n) in H there exists a sequence $(\tilde{f}_n)$, with $\tilde{f}_n \in \mathrm{Co}(f_n, f_{n+1}, \dots)$ such that, for a.e. ω in Ω, the sequence $(\tilde{f}_n(\omega))$ converges weakly in X.*

Theorem B. *A subset A of $L^1(\mu, X)$ is rwc iff given any $\varepsilon > 0$ there exist an integer N and a rwc subset H of NW such that $A \subseteq H + \varepsilon B(0)$, where $B(0)$ is the unit ball of $L^1(\mu, X)$.*

INTRODUCTION

Let (Ω, Σ, μ) be a probability space, X an arbitrary Banach space, and $L^1(\mu, X)$ the Banach space of Bochner integrable functions $f: \Omega \to X$ equipped with its usual norm [6, p. 50]. The problem of characterizing the rwc (relatively weakly compact) subsets of the space $L^1(\mu, X)$ is a well-known long-standing open problem, see Chapter IV of [6] for a review of known results about this problem up to 1977 and [1, 2, 3, 7, 9, 12] for some more recent results. In this note we present a characterization of the rwc subsets of the space $L^1(\mu, X)$. The characterization is obtained in two steps. In the first step we characterize the rwc subsets of the set $W = \{f \in L^1(\mu, X): \text{ for a.e. } \omega \text{ in } \Omega,\ \|f(\omega)\| \le 1\}$. This result is as follows: A subset H of W is rwc iff given any sequence (f_n) in H there exists a sequence $(\tilde{f}_n)$ with $\tilde{f}_n \in \mathrm{Co}(f_n, f_{n+1}, \dots)$ such that, for a.e. ω in Ω, the sequence $(\tilde{f}_n(\omega))$ converges weakly in X. In the second step we show that a subset A of $L^1(\mu, X)$ is rwc iff it is a small perturbation of a rwc subset of NW for some integer N. More precisely, we prove the following result: A subset A of the space $L^1(\mu, X)$ is rwc iff given any $\varepsilon > 0$ there exist

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an integer N and a rwc subset of H of NW such that $A \subseteq H + \varepsilon B(0)$, where $B(0)$ is the unit ball of $L^1(\mu, X)$. This paper also contains several corollaries of these results. The main ingredients of the proofs are an ad hoc lemma (Lemma 1) and the results established by M. Talagrand in his beautiful paper [12]. In §1 we have collected needed notation and preliminary results. The main results are in §2.

1. Notation and preliminaries

Our notation and terminology are quite standard and are in general those of [6]. All the Banach spaces used in this paper are taken to be defined on the field of real numbers $\mathbb{R}$. For a Banach space X, by X^* we denote its continuous dual and by $\langle x, x^* \rangle$ the natural duality between X and X^*. For a subset A of X, by $\operatorname{Co}(A)$ we denote the convex hull of A. By $B(0)$ we denote the closed unit ball of the Banach space under consideration. The context avoids misunderstanding. Throughout the paper (Ω, Σ, μ) is an arbitrary probability space, and X an arbitrary Banach space. The space of Bochner integrable functions $f: \Omega \to X$ is denoted by $L^1(\mu, X)$. When $X = \mathbb{R}$, we write $L^1(\mu)$ instead of $L^1(\mu, \mathbb{R})$. The space $L^1(\mu, X)$ is equipped with its usual norm $\|f\| = \int_\Omega \|f(\omega)\| d\mu(\omega)$. For an integer N we put $W(N) = \{f \in L^1(\mu, X)$: for a.e. ω in Ω, $\|f(\omega)\| \leq N\}$. When $N = 1$, the set $W(1)$ is simply denoted by W. Observe that $W(N) = NW$. For a subset H of W and ω in Ω we put $H(\omega) = \{f(\omega): f \in H\}$. Strictly speaking, $H(\omega)$ is not well defined since the elements of H are not single functions but a class of functions. To make the definition of $H(\omega)$ precise, one can introduce a lifting ρ of $L^\infty(\mu)$, and define $\rho(f)$ as in [8, p. 212] or [10, p. 76], and put $H(\omega) = \{\rho(f)(\omega): f \in H\}$. However, not to overcomplicate the notations, we do not introduce a lifting but deal with the elements of W as if they are strongly measurable bounded single functions. Finally, we recall the following result [4, p. 227].

Lemma (A. Grothendieck). *A subset A of a Banach space Y is rwc iff given any $\varepsilon > 0$ there exists an rwc set H_ε such that $A \subseteq H_\varepsilon + \varepsilon B(0)$.*

2. Weakly compact subsets of $L^1(\mu, X)$

A key result of this paper is the following lemma, which has turned out to be quite useful in connection with Talagrand's results in [12].

Lemma 1. *Let A be a bounded subset of a Banach space Y. Then A is rwc iff given any sequence (y_n) in A, there exists a sequence $(\tilde{y}_n)$ with $\tilde{y}_n \in \operatorname{Co}(y_n, y_{n+1}, \dots)$ that converges weakly.*

Proof. Assume A is rwc. Then, any sequence (y_n) in A has a weakly convergent subsequence (y_{n_k}). Put $\tilde{y}_k = y_{n_k}$. Then clearly $\tilde{y}_k \in \operatorname{Co}(y_k, y_{k+1}, \dots)$, and the sequence $(\tilde{y}_k)$ converges weakly. To prove the converse, it is enough, by James's Theorem [11], to show that every nontrivial functional y^* in Y^* attains its maximum on $\overline{\operatorname{Co}}(A)$. To this end, let y^* be a nontrivial functional in

Y^* and $\beta = \sup_{y\in\overline{\mathrm{Co}}(A)}\langle y, y^*\rangle$. Since $\beta = \sup_{y\in A}\langle y, y^*\rangle$ as well, there exists a sequence (y_n) in A such that $\beta = \lim\langle y_n, y^*\rangle$, as $n \to \infty$. By hypothesis, there exists a sequence $(\tilde{y}_n)$, with $\tilde{y}_n \in \mathrm{Co}(y_n, y_{n+1}, \dots)$, that converges weakly to an element, say a, of $\overline{\mathrm{Co}}(A)$. Each $\tilde{y}_n$ is of the form

$$\tilde{y}_n = \sum_{i\le k_n} \lambda_i y_{n+i}, \qquad \text{with } \lambda_i \ge 0 \text{ and } \sum_{i\le k_n}\lambda_i = 1.$$

Let $\varepsilon > 0$ be arbitrary. Since $\beta = \lim\langle y_n, y^*\rangle$, there exists an integer N such that, for all $n \ge N$, $\langle y_n, y^*\rangle > \beta - \varepsilon$. So for all $n \ge N$,

$$\langle \tilde{y}_n, y^*\rangle = \sum_{i\le k_n}\lambda_i\langle y_{n+i}, y^*\rangle > \sum_{i\le k_n}\lambda_i(\beta-\varepsilon) = \beta - \varepsilon.$$

Hence, $\langle a, y^*\rangle \ge \beta - \varepsilon$. As $\beta = \sup_{y\in\overline{\mathrm{Co}}(A)}\langle y, y^*\rangle$, the element a belongs to $\overline{\mathrm{Co}}(A)$, and ε is arbitrary, we conclude that $\beta = \langle a, y^*\rangle$ and $\overline{\mathrm{Co}}(A)$ is weakly compact. Hence A is rwc.

As a corollary of this lemma we have the following result.

Corollary 2. *A bounded sequence (y_n) in a Banach space Y converges weakly to an element a in Y iff given any subsequence $z_k = y_{n_k}$ of (y_n) there exists a sequence $(\tilde{z}_k)$, with $\tilde{z}_k \in \mathrm{Co}(z_k, z_{k+1}, \dots)$, that converges weakly to a.*

Proof. The implication $(\Rightarrow)$ is clear. On the other hand, if the condition of the corollary holds, by the preceding lemma, the sequence (y_n) is rwc and a is the unique weak cluster point of (y_n). Hence $y_n \to a$ weakly.

Another result we need for the proof of the main theorem is the following lemma, which is of independent interest. We recall that $W = \{f \in L^1(\mu, X)$: for a.e. ω in Ω, $\|f(\omega)\| \le 1\}$.

Lemma 3. *Let (f_n) be a sequence in W. Assume that*

(a) *the sequence (f_n) converges weakly to a function f in $L^1(\mu, X)$; and*

(b) *for a.e. ω in Ω, the sequence $(f_n(\omega))$ is weakly Cauchy. Then, for a.e. ω in Ω, $f_n(\omega) \to f(\omega)$ weakly in X.*

Proof. Let E be a negligible set in Σ such that, for each ω in $\Omega \setminus E$, the sequence $(f_n(\omega))$ is weakly Cauchy. For ω in $\Omega\setminus E$ and x^* in X^*, let

$$h_x * (\omega) = \lim_{n\to\infty}\langle f_n(\omega), x^*\rangle. \tag{1}$$

Since the sequence (f_n) converges weakly to f, there exists a sequence $(\tilde{f}_n)$, with $\tilde{f}_n \in \mathrm{Co}(f_n, f_{n+1}, \dots)$, such that

$$\|\tilde{f}_n - f\| = \int_\Omega \|\tilde{f}_n(\omega) - f(\omega)\|d\mu(\omega) \to 0, \qquad \text{as } n\to\infty. \tag{2}$$

Each $\tilde{f}_n$ is of the form

$$\tilde{f}_n = \sum_{i\le k_n}\lambda_i f_{n+i}, \qquad \text{with } \lambda_i \ge 0 \text{ and } \sum_{i\le k_n}\lambda_i = 1.$$

Using (1) and this expression of $\tilde{f}_n$, one can easily see that for each ω in $\Omega\backslash E$ and x^* in X^*,

$$\lim_{n\to\infty}\langle\tilde{f}_n(\omega), x^*\rangle = h_x * (\omega), \tag{3}$$

too. Now by (2), there exist a negligible set F in Σ and integers $n_1 < n_2 < \cdots < n_k < \cdots$ such that, for each ω in $\Omega\backslash F$, $\|\tilde{f}_{n_k}(\omega)-f(\omega)\| \to 0$ as $k\to\infty$. In particular, $\tilde{f}_{n_k}(\omega) \to f(\omega)$ weakly. Hence by (3) and (1), we conclude that for each ω in $\Omega\setminus(E\cup F)$, $f_n(\omega)\to f(\omega)$ weakly.

The main result of this paper is the following theorem.

Theorem 4. *A subset H of W is rwc iff given any sequence (f_n) in H there exists a sequence $(\tilde{f}_n)$, with $\tilde{f}_n \in \mathrm{Co}(f_n, f_{n+1}, \dots)$, such that for a.e. ω in Ω the sequence $(\tilde{f}_n(\omega))$ converges weakly in X.*

Proof. Necessity. Assume H is rwc, and let (f_n) be an arbitrary sequence in H. Then by Talagrand's Theorem 1 [12], there exists a sequence $(\tilde{f}_n)$, with $\tilde{f}_n \in \mathrm{Co}(f_n, f_{n+1}, \dots)$, and two sets C, L in Σ with $\mu(C\cup L) = 1$, such that

(a) for each ω in C the sequence $(\tilde{f}_n(\omega))$ is weakly Cauchy; and,

(b) for each ω in L there exists an integer k such that the sequence $(\tilde{f}_n(\omega))_{n\geq k}$ is equivalent to the unit basis of l^1.

Should the measure of the set L be strictly positive, by Talagrand's Lemma 4, for some integer k, the sequence $(\tilde{f}_n)_{n\geq k}$ would be equivalent to the unit basis of l^1. Since the sequence $(\tilde{f}_n)$ lies in the set $\overline{\mathrm{Co}}(H)$, which is weakly compact (Krein–Smulyan's Theorem), this is not possible. Therefore $\mu(L) = 0$, $\mu(C) = 1$, and for a.e. ω in Ω, the sequence $(\tilde{f}_n(\omega))$ is weakly Cauchy. Then by Talagrand's Lemma 8, the sequence $(\tilde{f}_n)$ is weakly Cauchy in $L^1(\mu, X)$. Since the sequence $(\tilde{f}_n)$ lies in the weakly compact set $\overline{\mathrm{Co}}(H)$, it has a weak cluster point f and, being weakly Cauchy, $\tilde{f}_n \to f$ weakly. Now the preceding lemma shows that, for a.e. ω in Ω, $\tilde{f}_n(\omega)\to f(\omega)$ weakly.

Sufficiency. Assume the condition of the theorem holds. Let (f_n) be an arbitrary sequence in H. Then by hypothesis, there exists a sequence $(\tilde{f}_n)$ with $\tilde{f}_n \in \mathrm{Co}(f_n, f_{n+1}, \dots)$ such that, for a.e. ω in Ω, the sequence $(\tilde{f}_n(\omega))$ converges weakly in X. Let $\tilde{f}(\omega) = \text{weak}\ \lim \tilde{f}_n(\omega)$ whenever this limit exists and $\tilde{f}(\omega) = 0$ otherwise. Since $\|f_n(\omega)\| \leq 1$ for a.e. ω in Ω and all n in $\mathbb{N}$, we have $\|f(\omega)\| \leq 1$ for a.e. ω in Ω. Also, the functions $\tilde{f}_n$ being strongly measurable, f is "almost separably" valued and weakly measurable. Hence, by Pettis's Measurability Theorem [6, p. 42] f is strongly measurable, and $f\in W$. Then, by Talagrand's Lemma 8, $\tilde{f}_n \to f$ weakly. Hence by Lemma 1, H is rwc.

We proceed with some corollaries of this theorem. The first corollary is an immediate consequence of Theorem 4 and Corollary 2.

Corollary 5. *A sequence* (f_n) *in* W *converges weakly to a function* f *in* $L^1(\mu, X)$ *iff, given any subsequence* $g_k = f_{n_k}$ *of* (f_n), *there exists a sequence* $(\tilde{g}_k)$ *with* $\tilde{g}_k \in \operatorname{Co}(g_k, g_{k+1}, \dots)$ *such that for a.e.* ω *in* Ω, $\tilde{g}_k(\omega) \to f(\omega)$ *weakly in* X.

The most useful corollary of the theorem seems to be the following one.

Corollary 6. *Let* H *be a subset of* W *and assume that for a.e.* ω *in* Ω, *the set* $H(\omega) = \{f(\omega): f \in H\}$ *is rwc. Then* H *is rwc.*

Proof. Let (f_n) be an arbitrary sequence in H. As in the proof of Theorem 4, by Talagrand's Theorem 1, there exist a sequence $(\tilde{f}_n)$ with $\tilde{f}_n \in \operatorname{Co}(f_n, f_{n+1}, \dots)$ and two sets C, L in Σ with $\mu(C \cup L) = 1$ such that

(a) for each ω in C the sequence $(\tilde{f}_n(\omega))$ is weakly Cauchy; and,

(b) for each ω in L there exists an integer k such that the sequence $(\tilde{f}_n(\omega))_{n \geq k}$ is equivalent to the unit basis of l^1.

Since, for a.e. ω in Ω, the set $\overline{\operatorname{Co}} H(\omega)$ is rwc and the sequence $(\tilde{f}_n(\omega))$ lies in this set, we conclude that $\mu(L) = 0$, $\mu(C) = 1$, and for a.e. ω in Ω, the sequence $(\tilde{f}_n(\omega))$ converges weakly in X. Hence by the theorem, the set H is rwc.

As an immediate application of this corollary we have the following result, which is due to J. Diestel [5].

Proposition 7. *Let* K *be a weakly compact subset of* X *and* $\widetilde{K} = \{f \in L^1(\mu, X):$ *for a.e.* ω *in* Ω, $f(\omega) \in K\}$. *Then* $\widetilde{K}$ *is rwc.*

Proof. We can assume that K is contained in the unit ball of X. Then $\widetilde{K}$ is a subset of W and the preceding corollary applies to $\widetilde{K}$.

For the proof of the next theorem, we recall that a subset A of $L^1(\mu, X)$ is said to be uniformly integrable if the set $V(A) = \{\|f(\cdot)\|: f \in A\}$ is uniformly integrable [6, p. 74].

Theorem 8. *A subset* A *of* $L^1(\mu, X)$ *is rwc iff given* $\varepsilon > 0$ *there exist an integer* N *and a rwc subset* H *of* $W(N)$ *such that* $A \subseteq H + \varepsilon B(0)$.

Proof. Assume A is rwc. Then A is uniformly integrable. This follows, for instance, from [6, IV.2.4]. So we have:

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall B \in \Sigma \left(\mu(B) < \delta \Rightarrow \sup_{f \in A} \int_B \|f\| d\mu < \varepsilon \right). \tag{4}$$

Let $\varepsilon > 0$ be fixed. Then using (4), we can find an integer N such that for each f in A, $\mu(\{\omega \in \Omega: \|f(\omega)\| > N\}) < \delta$. Now if for f in A, we put $f_N = f \cdot \chi_{E_f}$, where $E_f = \{\omega \in \Omega: \|f(\omega)\| \leq N\}$, we get that $\|f - f_N\| < \varepsilon$ for all f in A. Hence if we put $H = \{f_N: f \in A\}$, then $H \subseteq W(N)$ and $A \subseteq H + \varepsilon B(0)$. The set H is contained in the set $D = \{f \cdot \chi_E: f \in A,\ E \in \Sigma\}$. The sets A and $\{\chi_E: E \in \Sigma\}$ being rwc in $L^1(\mu, X)$ and $L^1(\mu)$, respectively, one can easily see that the set D is rwc. Hence the set H is rwc. This proves the necessary part of the theorem. The sufficiency part follows from the lemma of A. Grothendieck recalled as in §1.

The following corollary of this theorem gives a useful sufficient condition for a subset A of $L^1(\mu, X)$ to be rwc, also see Corollary 12 of [12].

Corollary 9. *Let A be a bounded uniformly integrable subset of $L^1(\mu, X)$. Assume that for a.e. ω in Ω, the set $A(\omega) = \{f(\omega): f \in A\}$ is rwc. Then A is rwc.*

Proof. Let $\varepsilon > 0$ be an arbitrary. With the notations of the proof of the preceding theorem, $A \subseteq H + \varepsilon B(0)$. The set H is contained in $W(N)$ for some integer N, and for a.e. ω in Ω, the set $H(\omega)$ is contained in the rwc set $A(\omega) \cup \{0\}$. Hence by Corollary 6, H is rwc. Then by the preceding theorem, A is rwc.

As an immediate application of this corollary, we mention the following well-known result [6, IV.2.1].

Proposition 10. *Assume X is reflexive. Then a subset A of $L^1(\mu, X)$ is rwc iff it is bounded and uniformly integrable.*

The final result of the paper is the following "Lebesgue's Dominated Convergence Theorem"-type result. This result is also an immediate consequence of Corollary 9.

Proposition 11. *Let $g: \Omega \to \mathbb{R}$ be a positive integrable function and (f_n) a sequence in $L^1(\mu, X)$ such that*

(a) *for a.e. ω in Ω and all $n \in \mathbb{N}$, $\|f_n(\omega)\| \le g(\omega)$; and,*

(b) *for a.e. ω in Ω, the sequence $(f_n(\omega))$ is rwc. Then the sequence (f_n) is rwc.*

Added in proof. Prof. P. Saab has informed us that the set D in the proof of Theorem 8 need not be rwc, and she is right. However this does not affect the subsequent results since they depend only on the sufficiency part of this theorem. Moreover we have the following result which is more general than Theorem 4.

Theorem (AB). *A subset A of $L^1(\mu, X)$ is rwc iff it is bounded, uniformly integrable and for any sequence (f_n) in A there exists a sequence $(\tilde{f}_n)$, with $\tilde{f}_n \in \mathrm{Co}(f_n, f_{n+1}, \dots)$, such that for a.e. ω in Ω the sequence $(\tilde{f}_n(\omega))$ converges weakly in X.*

Sketch of the proof. The proof of the necessity is very similar to that of Theorem 4. However one should use Lemma 7 of [12] instead of Theorem 1 of [12]. For the sufficiency, let A, (f_n), and $(\tilde{f}_n)$ be as in the statement. Let $f(\omega) = \text{weak-}\lim \tilde{f}_n(\omega)$ whenever this limit exists and $f(\omega) = 0$ otherwise. Then f is strongly measurable and since $\|f(\omega)\| \le \liminf \|\tilde{f}_n(\omega)\|$ a.e., f

is in $L^1(\mu, X)$. Let $h\colon \Omega \to X^*$ be a weak* scalarly measurable essentially bounded function, i.e., $h \in L^1(\mu, X)^*$. Then, A being uniformly integrable, the sequence $(\langle \tilde{f}_n, h\rangle)$, is rwc. Since, for a.e. ω in Ω, $\langle \tilde{f}_n(\omega), h(\omega)\rangle \to \langle f(\omega), h(\omega)\rangle$, we conclude that the sequence $(\langle \tilde{f}_n, h\rangle)$ converges to $\langle f, h\rangle$ in $L^1(\mu)$. Hence $\tilde{f}_n \to f$ weakly in $L^1(\mu, X)$ and, by Lemma 1, A is rwc.

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Department of Mathematics, Bogazici University, 80815 Bebek Istanbul, Turkey