SPLIT BRAIDS

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Abstract. Let $B_n$ be the group of braids on $n$ strings with standard generators $\sigma_1, \ldots, \sigma_{n-1}$. For $i \in \{1, 2, \ldots, n-1\}$ we let $B'_n$ be the subgroup of $B_n$ generated by the elements $\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_{n-1}$. In this paper we give an algorithm for deciding if, given $\alpha \in B_n$ there is $i \in \{1, 2, \ldots, n-1\}$ such that $\alpha$ is conjugate into $B'_n$. We call such a braid a split braid. Such a split braid gives rise to a split link. This algorithm gives a partial solution to the problem of finding braids that represent reducible mapping classes. It also represents a contribution to the algebraic link problem and it gives a way of determining if a braid in $B_n$ can be conjugated into the subgroup $B_{n-1}$, which we identify with $B_{n-1}$.

1. Introduction

For $n > 1$ let $B_n$ be the group of braids on $n$ strings. Then $B_n$ has a presentation as a group with generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if} \quad 1 \leq i, j \leq n-1 \quad \text{and} \quad |i-j| > 1;$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad \text{for} \quad 1 \leq i < n-1.$$

It is also well known that $B_n$ has a faithful representation in $\text{Aut}(F(n))$, the group of automorphisms of the free group $F(n)$ of rank $n$. If $x_1, \ldots, x_n$ are fixed free generators for $F(n)$, then the action of $B_n$ on $F(n)$ is given by the following actions of the generators $\sigma_1, \ldots, \sigma_{n-1}$ of $B_n$ on the generators $x_1, \ldots, x_n$ of $F(n)$:

$$\sigma_i(x_j) = \begin{cases} x_j & \text{if} \quad j \neq i, \ i+1, \\ x_{i+1} & \text{if} \quad j = i, \\ x_{i+1}^{-1}x_ix_{i+1} & \text{if} \quad j = i+1. \end{cases}$$

It is easy to check that with this action the word $x_1x_2\cdots x_n$ is fixed and that if $\alpha \in B_n$, then each $\alpha(x_i)$ is a conjugate of some $x_j$. In fact these two properties characterize the image of or braid group in $\text{Aut}(F(n))$ and we
sometimes confuse a braid $\alpha$ with the $n$–tuple of words $(\alpha(x_1), \ldots, \alpha(x_n))$. This representation of the braid groups has a geometric interpretation since $B_n$ acts as a group of mapping classes of an $n$–punctured disc $D_n$. The above action can then be realized as the action of $B_n$ on the fundamental group of $D_n$, which is a free group of rank $n$. We identify this group with $F(n)$. Let $\pi_n: B_n \to S_n$ be the permutation representation of $B_n$ onto the symmetric group. For more information on braid groups see [Bi, Ma].

For $i \in \{1, 2, \ldots, n-1\}$ we let $B'_n$ be the subgroup of $B_n$ generated by the elements $\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_{n-1}$. In this paper we give an algorithm for determining if, given $\alpha \in B_n$, there is $i \in \{1, 2, \ldots, n-1\}$ such that $\alpha$ is conjugate into $B'_n$. We call such a braid a split braid. This algorithm gives a partial solution to the problem of finding braids that represent reducible mapping classes, i.e. braids whose action on the disc $D_n$ fixes the isotopy class of a nontrivial, nonboundary parallel simple closed curve on $D_n$. Furthermore the closed braid $\hat{\alpha}$ corresponding to a split braid $\alpha$ is a split link in the sense that there is an $S^2$ disjoint from $\hat{\alpha}$ separating the components of $\hat{\alpha}$. Thus our result also represents a contribution to the algebraic link problem. Also note that in the case $i = n - 1$ our method gives a way of determining if a braid in $B_n$ can be conjugated into the subgroup $B'_n$, which we can identify with $B_{n-1}$.

2. The algorithm

For each subset $I \subset \{1, 2, \ldots, n\}$ there is a map $\varphi_I: B_n \to B_n$, which consists of taking a braid $\alpha$ in $B_n$ and for each $i \in I$ pulling out the string that ends up in the $i$th position after doing $\alpha$, and putting it back in as a straight string to the right of all the other strings. We think of $\varphi_I(\alpha)$ as belonging to the subgroup $B_n$ consisting of braids fixing the last $|I|$ strings, where $|I|$ is the cardinality of the set $I$. This process involves a relabelling of the strings and is illustrated in the case $n = 4$, $I = \{3\}$ in Figure 1.

Note that the map $\varphi_I$ is not a homomorphism; however, we do have the following result:

**Lemma 2.1.** Let $I \subset \{1, 2, \ldots, n\}$ and $\alpha, \beta \in B_n$. Then $\varphi_I(\alpha\beta) = \varphi_I(\alpha)\varphi_I(\beta)$, where $J = \pi(\alpha^{-1})(I)$.

![Figure 1](https://www.ams.org/journal-terms-of-use)
Proof. The proof consists of noticing that if $i \in I$ and if $j = \pi(\alpha^{-1})(i)$, then it is the $j$th string of $\beta$ that is removed when we remove the $i$th string of $\alpha$. □

For $I \subset \{1, 2, \ldots, n\}$ we let $Y_I$ be the stabilizer of $I$ in $B_n$, i.e. the subgroup consisting of all the braids $\alpha$ such that $\pi(\alpha^{-1})(I) = I$. Since $Y_I$ contains the pure braid group it follows easily that $Y_I$ is of finite index in $B_n$.

Then the key observation is:

**Proposition 2.2.** Let $\alpha, \gamma \in Y_I$ and $\beta = \gamma \alpha \gamma^{-1}$. Then $\varphi_I(\beta)$ is a conjugate of $\alpha$ in $B_n$.

**Proof.** Let $\alpha$, $\gamma$, and $\beta$ be as above. Then Lemma 2.1 shows that

$$
\varphi_I(\beta) = \varphi_I(\gamma \alpha \gamma^{-1}) = \varphi_I(\gamma) \varphi_I(\alpha) \varphi_I(\gamma^{-1}),
$$

and so the result follows if we can show that $\varphi_I(\gamma^{-1}) = \varphi_I(\gamma)^{-1}$. Again Lemma 2.1 shows that we have, $\text{id} = \varphi_I(\gamma \gamma^{-1}) = \varphi_I(\gamma) \varphi_I(\gamma^{-1})$ and so the result follows. □

We next show that the definition of $\varphi_I$ amounts to the following algebraic operation as far as the braid automorphisms are concerned. If $m \leq n$, then we think of $F(m)$ as the subgroup of $F(n)$ generated by $x_1, \ldots, x_m$. We also use the notation $\alpha^*(I)$ instead of $\pi(\alpha^{-1})(I)$. Note that we have

$$(\alpha \beta)^*(I) = \pi((\alpha \beta)^{-1})(I) = \pi(\beta^{-1}) \pi(\alpha^{-1})(I) = \beta^* \alpha^*(I).$$

The next result is a consequence of Lemmas 2.4 and 2.5.

**Proposition 2.3.** Suppose that $I \subset \{1, 2, \ldots, n\}$. If $j \in \{1, 2, \ldots, n\} \setminus I$ then we let $n(j) = n(j, I)$ be the number of elements of $I$ that are strictly less than $j$. If $\alpha \in B_n$, then $\varphi_I(\alpha)(x_j) = \Sigma_I(\alpha(x_j))$ where $\Sigma_I : F(n) \to F(n - |I|)$ is the epimorphism defined by the following action on the generators of $F(n)$:

$$
\begin{align*}
\Sigma_I(x_k) &= x_{k-n(k)} & \text{if } k \in \{1, 2, \ldots, n\} \setminus I; \\
\Sigma_I(x_i) &= \text{identity} & \text{if } i \in I.
\end{align*}
$$

We let $\Sigma_I$ act on the generators of $B_n$ in the following way:

$$
\begin{align*}
\Sigma_I(\sigma_k) &= \sigma_{k-n(k)} & \text{if } k \in \{1, 2, \ldots, n-1\} \setminus I; \\
\Sigma_I(\sigma_i) &= \text{identity} & \text{if } i \in I.
\end{align*}
$$

In what follows it is convenient to define $n(i) = \infty$ if $i \in I$ and to have the convention that $x_\infty = \text{id}$ in $F(n)$, $\sigma_\infty = \text{id}$ in $B_n$.

**Lemma 2.4.** Let $X \in F(n)$ and $\varepsilon = \pm 1$.

(a) If $k, k+1 \in \{1, 2, \ldots, n\} \setminus I$, then $\Sigma_I(\sigma_k^\varepsilon(X)) = \sigma_{k-n(k)}(\Sigma_I(X))$.

(b) If $k \in I$, but $k+1 \in \{1, \ldots, n\} \setminus I$, then $\Sigma_I(\sigma_k^\varepsilon(X)) = \Sigma_{I'}(X)$ where $I' = \sigma_k^*(I)$.
(c) If \( k \) does not belong to \( I \) but \( k + 1 \) does, then \( \Sigma_I(\sigma_k^e(X)) = \Sigma_{I'}(X) \)
where \( I' = \sigma_k^e(I) \).

(d) If \( k, k + 1 \in I \), then \( \Sigma_I(\sigma_k^e(X)) = \Sigma_I(X) \).

The functions \( \Sigma_I \) satisfy
\[
\Sigma_I(\sigma_k^eX) = \Sigma_{I \subseteq I'}(\sigma_k^e)\Sigma_{I'}(X),
\]
where \( I' = \sigma_k^e(I) \) and \( X \in F(n) \).

Proof. In each case below we only prove the result for \( e = +1 \), the other case
being similar. (a) Suppose that \( k, k + 1 \in \{1, 2, \ldots, n\} \setminus I \). Then \( n(k) = n(k + 1) \) and so we have

\[\begin{align*}
(\text{i}) & \quad \Sigma_I(\sigma_k(x_k)) = \Sigma_I(x_{k+1}) = x_{k-n(k)+1} = \sigma_{k-n(k)}(x_{k-n(k)}) = \sigma_{k-n(k)}(\Sigma_I(x_k)); \\
(\text{ii}) & \quad \Sigma_I(\sigma_k(x_{k+1})) = \Sigma_I(x_{k+1}^{-1}x_kx_{k+1}) = x_{k+1-n(k)}x_{k-n(k)}x_{k+1-n(k)} = \\
& \quad \sigma_{k-n(k)}(\Sigma_I(x_{k+1})); \text{ and} \\
(\text{iii}) & \quad \text{If } j \neq k, k + 1, \text{ then } \Sigma_I(\sigma_k(x_j)) = \Sigma_I(x_j) = x_{j-n(j)} \quad \text{and} \\
& \quad \sigma_{k-n(k)}(\Sigma_I(x_j)) = \sigma_{k-n(k)}(x_j). \quad \text{To see that } x_{j-n(j)} = \sigma_{k-n(k)}(x_j), \text{ we consider case 1: } j < k, \text{ and case} \\
& \quad 2: j > k + 1. \text{ In case 1 we have two subcases; (a) } j \in I; \text{ and (b) } j \text{ not in } I. \text{ If} \\
& \quad (a) \text{ then } x_{j-n(j)} = \text{id and we are O.K. If (b), then } n(k) - n(j) \leq k - j - 1 < k - j \text{ showing that } k - n(k) > j - n(j), \text{ which gives the result. If we have case 2,} \\
& \quad \text{then again we consider the two cases (a) and (b) above. If we have (a), then} \\
& \quad x_{j-n(j)} = \text{id and we are done. If (b), then } k, k + 1 \text{ not in } I \text{ means that} \\
& \quad n(j) - n(k) \leq j - k - 2 \text{ and so } j - n(j) > k - n(k) + 1, \text{ which gives the result.} \text{ This proves part (a) of Lemma 2.4.} \\
\end{align*}\]

(b) Suppose that \( k \in I \), but \( k + 1 \in \{1, \ldots, n\} \setminus I \). Let \( I' = \sigma_k^e(I) = \\
(\{I \setminus \{k\}\} \cup \{k + 1\}) \). Then (i) \( \Sigma_I(\sigma_k(x_k)) = \Sigma_I(x_{k+1}) = x_{k+1-n(k+1)} \) and \( \Sigma_{I'}(x_k) = \\
x_{k-n(k+1)} \); but it is easily seen that \( n(k, I') = n(k+1, I) \), and so the result
follows in this case. (ii) We have \( \Sigma_I(\sigma_k(x_{k+1})) = \Sigma_I(x_{k+1}^{-1}x_kx_{k+1}) = \text{id} \) and \\
\( \Sigma_{I'}(x_{k+1}) = \text{id}. \) (iii) If \( j \neq k, k + 1 \), then \( \Sigma_I(\sigma_k(x_j)) = \Sigma_I(x_j) = x_{j-n(j)} \), \\
and \( \Sigma_{I'}(x_j) = x_{j-n(j)} \). \text{ The result now follows from the fact that } n(j, I) = \\
n(j, I') \text{ if } j \neq k, k + 1.

(c) Suppose that \( k + 1 \in I \), but \( k \in \{1, \ldots, n\} \setminus I \). Let \( I' = \sigma_k^e(I) = \\
(\{I \setminus \{k\}\} \cup \{k + 1\}) \). Then (i) \( \Sigma_I(\sigma_k(x_k)) = \Sigma_I(x_{k+1}) = \text{id} \) and \( \Sigma_{I'}(x_k) = \text{id}. \) (ii) \\
(\text{ii}) \text{ We have } \Sigma_I(\sigma_k(x_{k+1})) = \Sigma_I(x_{k+1}^{-1}x_kx_{k+1}) = \text{id} \quad \text{and} \\
\Sigma_{I'}(x_{k+1}) = \text{id}. \quad \text{ (iii) If } j \neq k, k + 1, \text{ then } \Sigma_I(\sigma_k(x_j)) = \Sigma_I(x_j) = x_{j-n(j)} \text{ and } \Sigma_{I'}(x_j) = x_{j-n(j)} \text{.} \text{ The result now follows since } n(j, I) = n(j, I') \text{ if } j \neq k, k + 1.

(d) If \( k, k + 1 \in I \), then \( \Sigma_I(\sigma_k^e(x_j)) = x_{j-n(j)} = \Sigma_I(x_j) \) for all \( j \).

The remaining formula is easily checked using parts (a), (b), (c), and (d), 
which cover all cases. \( \square \)

In light of the above result we are now justified in writing 
\( \Sigma_I\sigma_k^e \alpha = \Sigma_{I \cup I'}(\sigma_k^e)\Sigma_{I'}\alpha \) where \( \alpha \in B^n \). We do this in what follows.
The idea for the proof of Proposition 2.3 is to show that $\Sigma_I$ has the same properties as does $\varphi_I$; so we next prove the analogue of Lemma 2.1:

**Lemma 2.5.** Let $\alpha, \beta \in B_n$. Then $\Sigma_I(\alpha\beta) = \Sigma_I(\alpha)\Sigma_I(\beta)$, where $J = \alpha^*(I)$.

*Proof.* We induct on the length $r$ of the word $\alpha$. Clearly the result is true in the case $r = 0$ ($\alpha = \text{identity}$). So now suppose that the result is true for all words $\alpha$ of length $N \geq 0$ and suppose that $\alpha = \sigma_k^\varepsilon \alpha'$, where $\varepsilon = \pm 1$ and the length of $\alpha'$ is $N$. By induction we now have $\Sigma_I(\alpha'\beta) = \Sigma_I(\alpha')\Sigma_I(\beta)$ for all $I \subset \{1, \ldots, n\}$, with $J = (\alpha')^*(I)$. Thus if $I \subset \{1, \ldots, n\}$, $I' = (\sigma_k^\varepsilon)^*(I)$ and $J' = (\alpha')^*(I')$, then by Lemma 2.4 we have

$$\Sigma_I(\alpha\beta) = \Sigma_I(\sigma_k^\varepsilon \alpha' \beta) = \Sigma_I \cup I' (\sigma_k^\varepsilon) \Sigma_I' (\alpha' \beta)$$

$$= \Sigma_I \cup I' (\sigma_k^\varepsilon) \Sigma_I' (\alpha') \Sigma_J (\beta)$$

$$= \Sigma_I (\sigma_k^\varepsilon \alpha') \Sigma_J (\beta),$$

which is what we require since

$$J' = (\alpha')^*(I') = (\alpha')^*((\sigma_k^\varepsilon)^*(I)) = (\sigma_k^\varepsilon \alpha')^*(I). \quad \square$$

Now Proposition 2.3 follows from the last result and the fact that $\varphi_I \sigma_k^\varepsilon = \Sigma_I (\sigma_k^\varepsilon)$ for all $I \subset \{1, \ldots, n\}$, $\varepsilon = \pm 1$, and $k = 1, \ldots, n - 1$. \quad \square

**Lemma 2.6.** Let $\{y_1, \ldots, y_m\}$ be a set of coset representatives for $Y_I$ in $B_n$. Let $J = \{1, 2, \ldots, i\}$ and $I = \{i + 1, i + 2, \ldots, n\}$. If $y \in B_n$ and $a = axa_2 \in B'^n$, where $ax \in \langle a_1, \ldots, a_{i-1} \rangle$ and $a_2 \in \langle a_{i+1}, \ldots, a_{n-1} \rangle$, then there is $k = 1, \ldots, m$ such that $\varphi_I(y_k y_\gamma^{-1} y_k^{-1})$ is a conjugate of $a_1$ in $B_n$ and $\varphi_J(y_k y_\gamma^{-1} y_k^{-1})$ is a conjugate of $a_2$ in $B_n$.

*Proof.* For this choice of $\gamma$ there is $k = 1, \ldots, m(i)$ such that $y_k \gamma$ is in $Y_I$. Then by Lemma 2.1 we see that

$$\varphi_I(y_k \gamma \alpha \gamma^{-1} y_k^{-1}) = \varphi_I(y_k \gamma) \varphi_I(\alpha \gamma^{-1} y_k^{-1})$$

$$= \varphi_I(y_k \gamma) \varphi_I(\alpha) \varphi_I(\gamma^{-1} y_k^{-1})$$

$$= \varphi_I(y_k \gamma) \varphi_I(\alpha_1 \alpha_2) \varphi_I(\gamma y_k)^{-1}$$

$$= \varphi_I(y_k \gamma) \alpha_1 \varphi_I(\gamma y_k)^{-1}$$

is a conjugate of $a_1$ in $B_n$ as required. A similar argument shows that $\varphi_J(y_k \gamma \alpha \gamma^{-1} y_k^{-1})$ is a conjugate of $\varphi_J(\alpha_2)$; but one easily sees that $\varphi_J(\alpha_2)$ is a conjugate of $\varphi_J(\alpha_2)$ in $B_n$ and so the result follows. \quad \square

For the moment let us keep the notation of Lemma 2.6. Now if $m \geq i + j - 1$, then we let $\Pi_i : B_j \to B_m$ be the monomorphism determined by $\Pi_i (\sigma_k) = \sigma_{k+i-1}$ for all $k = 1, \ldots, j - 1$. Then $\Pi_{i+1} \Sigma_I (\Sigma_k) = \Sigma_h$ for all $h = i + 1, \ldots, n - 1$. This fact, together with the proof of Lemma 2.6, gives the following result.
Corollary 2.7. If $\alpha = \alpha_1 \alpha_2 \in B_n^i$, where $\alpha_1 \in \langle \sigma_1, \ldots, \sigma_{i-1} \rangle$ and $\alpha_2 \in \langle \sigma_{i+1}, \ldots, \sigma_{n-1} \rangle$, then $\alpha$ is conjugate in $B_n$ to

$$(\varphi_j(y_k \gamma \alpha \gamma^{-1} y_k^{-1}))(\Pi_{i+1} \varphi_j(y_k \gamma \alpha \gamma^{-1} y_k^{-1})).$$

□

We now describe the algorithm for determining if we can conjugate $\alpha \in B_n$ into some $B_n^i$. This is based on the above results and the fact that the conjugacy problem is solved in $B_n^i$.

Step 1. For each $i = 1, 2, \ldots, n$ we let $I = \{1, 2, \ldots, n\}\{i\}$ and find a set $\{y_1, \ldots, y_{m(i)}\}$ of coset representatives for $Y_i$ in $B_n$.

Do Steps 2(i) and 3(i) for each $i = 1, \ldots, m(i)$. If $\alpha$ is conjugate to a braid in $B_n^i$, then we succeed for some value of $i$, by Corollary 2.7.

Step 2(i). Let $w_i = (\varphi_j(y_k \gamma \alpha \gamma^{-1} y_k^{-1}))(\Pi_{i+1} \varphi_j(y_k \gamma \alpha \gamma^{-1} y_k^{-1}))$ where we calculate this braid using Proposition 2.3. The braid $w_i$ has a representation as a product of the standard generators and their inverses, which is calculated using an algorithm described in the proof of a theorem of Artin (see [Bi, p. 30]).

Step 3(i). Use the solution of the conjugacy problem as presented in [Ga] (see also [Bi]) to determine whether $w_i$ and $\alpha$ are conjugate in $B_n^i$.

If no $w_i$ is conjugate to $\alpha$, for all $k = 1, \ldots, m(i)$, then $\alpha$ is not conjugate to a braid in $B_n^i$ by Corollary 2.7.

References


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