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SPLIT BRAIDS

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ABSTRACT. Let B_n be the group of braids on n strings with standard generators $\sigma_1, \dots, \sigma_{n-1}$. For $i \in \{1, 2, \dots, n-1\}$ we let B_n^i be the subgroup of B_n generated by the elements $\sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_{n-1}$. In this paper we give an algorithm for deciding if, given $\alpha \in B_n$ there is $i \in \{1, 2, \dots, n-1\}$ such that α is conjugate into B_n^i. We call such a braid a *split braid*. Such a split braid gives rise to a split link. This algorithm gives a partial solution to the problem of finding braids that represent reducible mapping classes. It also represents a contribution to the algebraic link problem and it gives a way of determining if a braid in B_n can be conjugated into the subgroup B_{n-1}, which we identify with B_{n-1}^{n-1}.

1. INTRODUCTION

For $n > 1$ let B_n be the group of braids on n strings. Then B_n has a presentation as a group with generators $\sigma_1, \dots, \sigma_{n-1}$ and relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{if } 1 \le i, j \le n-1 \text{ and } |i-j| > 1;$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \text{for } 1 \le i < n-1.$$

It is also well known that B_n has a faithful representation in $\mathrm{Aut}(F(n))$, the group of automorphisms of the free group $F(n)$ of rank n. If $x_1, \dots, x_n$ are fixed free generators for $F(n)$, then the action of B_n on $F(n)$ is given by the following actions of the generators $\sigma_1, \dots, \sigma_{n-1}$ of B_n on the generators $x_1, \dots, x_n$ of $F(n)$:

$$\sigma_i(x_j) = \begin{cases} x_j & \text{if } j \ne i,\ i+1, \\ x_{i+1} & \text{if } j = i, \\ x_{i+1}^{-1}x_ix_{i+1} & \text{if } j = i+1. \end{cases}$$

It is easy to check that with this action the word $x_1x_2\cdots x_n$ is fixed and that if $\alpha \in B_n$, then each $\alpha(x_i)$ is a conjugate of some x_j. In fact these two properties characterize the image of or braid group in $\mathrm{Aut}(F(n))$ and we

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sometimes confuse a braid α with the n–tuple of words $(\alpha(x_1), \dots, \alpha(x_n))$. This representation of the braid groups has a geometric interpretation since B_n acts as a group of mapping classes of an n–punctured disc D_n. The above action can then be realized as the action of B_n on the fundamental group of D_n, which is a free group of rank n. We identify this group with $F(n)$. Let π_n: $B_n \to S_n$ be the permutation representation of B_n onto the symmetric group. For more information on braid groups see [Bi, Ma].

For $i \in \{1, 2, \dots, n-1\}$ we let B_n^i be the subgroup of B_n generated by the elements $\sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_{n-1}$. In this paper we give an algorithm for determining if, given $\alpha \in B_n$, there is $i \in \{1, 2, \dots, n-1\}$ such that α is conjugate into B_n^i. We call such a braid a *split braid*. This algorithm gives a partial solution to the problem of finding braids that represent reducible mapping classes, i.e. braids whose action on the disc D_n fixes the isotopy class of a nontrivial, nonboundary parallel simple closed curve on D_n. Furthermore the closed braid $\hat{\alpha}$ corresponding to a split braid α is a split link in the sense that there is an S^2 disjoint from $\hat{\alpha}$ separating the components of $\hat{\alpha}$. Thus our result also represents a contribution to the algebraic link problem. Also note that in the case $i = n-1$ our method gives a way of determining if a braid in B_n can be conjugated into the subgroup B_n^i, which we can identify with B_{n-1}.

2. The algorithm

For each subset $I \subset \{1, 2, \dots, n\}$ there is a map φ_I: $B_n \to B_n$, which consists of taking a braid α in B_n and for each $i \in I$ pulling out the string that ends up in the ith position after doing α, and putting it back in as a straight string to the right of all the other strings. We think of $\varphi_I(\alpha)$ as belonging to the subgroup B_n consisting of braids fixing the last $|I|$ strings, where $|I|$ is the cardinality of the set I. This process involves a relabelling of the strings and is illustrated in the case $n = 4$, $I = \{3\}$ in Figure 1.

Note that the map φ_I is not a homomorphism; however, we do have the following result:

Lemma 2.1. *Let* $I \subset \{1, 2, \dots, n\}$ *and* $\alpha, \beta \in B_n$. *Then* $\varphi_I(\alpha\beta) = \varphi_I(\alpha)\varphi_J(\beta)$, *where* $J = \pi(\alpha^{-1})(I)$.

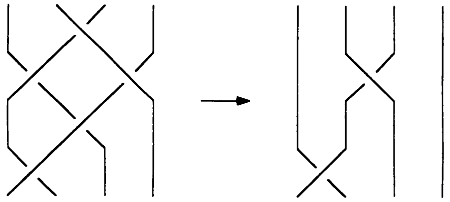

Figure 1

Proof. The proof consists of noticing that if $i \in I$ and if $j = \pi(\alpha^{-1})(i)$, then it is the jth string of β that is removed when we remove the ith string of α. □

For $I \subset \{1, 2, \dots, n\}$ we let Y_I be the stablizer of I in B_n, i.e. the subgroup consisting of all the braids α such that $\pi(\alpha^{-1})(I) = I$. Since Y_I contains the pure braid group it follows easily that Y_I is of finite index in B_n. Then the key observation is:

Proposition 2.2. *Let $\alpha, \gamma \in Y_I$ and $\beta = \gamma\alpha\gamma^{-1}$. Then $\varphi_I(\beta)$ is a conjugate of α in B_n.*

Proof. Let α, γ, and β be as above. Then Lemma 2.1 shows that

$$\varphi_I(\beta) = \varphi_I(\gamma\alpha\gamma^{-1}) = \varphi_I(\gamma)\varphi_I(\alpha)\varphi_I(\gamma^{-1}),$$

and so the result follows if we can show that $\varphi_I(\gamma^{-1}) = \varphi_I(\gamma)^{-1}$. Again Lemma 2.1 shows that we have, $\mathrm{id} = \varphi_I(\gamma\gamma^{-1}) = \varphi_I(\gamma)\varphi_I(\gamma^{-1})$ and so the result follows. □

We next show that the definition of φ_I amounts to the following algebraic operation as far as the braid automorphisms are concerned. If $m \le n$, then we think of $F(m)$ as the subgroup of $F(n)$ generated by $x_1, \dots, x_m$. We also use the notation $\alpha^*(I)$ instead of $\pi(\alpha^{-1})(I)$. Note that we have

$$(\alpha\beta)^*(I) = \pi((\alpha\beta)^{-1})(I) = \pi(\beta^{-1})\pi(\alpha^{-1})(I) = \beta^*\alpha^*(I).$$

The next result is a consequence of Lemmas 2.4 and 2.5.

Proposition 2.3. *Suppose that $I \subset \{1, 2, \dots, n\}$. If $j \in \{1, 2, \dots, n\}\backslash I$ then we let $n(j) = n(j, I)$ be the number of elements of I that are strictly less than j. If $\alpha \in B_n$, then $\varphi_I(\alpha)(x_j) = \Sigma_I(\alpha(x_j))$ where $\Sigma_I\colon F(n) \to F(n - |I|)$ is the epimorphism defined by the following action on the generators of $F(n)$:*

$$\Sigma_I(x_k) = x_{k-n(k)} \quad \text{if } k \in \{1, 2, \dots, n\}\backslash I;$$
$$\Sigma_I(x_i) = \text{identity} \quad \text{if } i \in I.$$

We let Σ_I act on the generators of B_n in the following way:

$$\Sigma_I(\sigma_k) = \sigma_{k-n(k)} \quad \text{if } k \in \{1, 2, \dots, n-1\}\backslash I;$$
$$\Sigma_I(\sigma_i) = \text{identity} \quad \text{if } i \in I.$$

In what follows it is convenient to define $n(i) = \infty$ if $i \in I$ and to have the convention that $x_{-\infty} = \mathrm{id}$ in $F(n)$, $\sigma_{-\infty} = \mathrm{id}$ in B_n.

Lemma 2.4. *Let $X \in F(n)$ and $\varepsilon = \pm 1$.*

(a) *If k, $k+1 \in \{1, 2, \dots, n\}\backslash I$, then $\Sigma_I(\sigma_k^\varepsilon(X)) = \sigma_{k-n(k)}^\varepsilon(\Sigma_I(X))$.*

(b) *If $k \in I$, but $k+1 \in \{1, \dots, n\}\backslash I$, then $\Sigma_I(\sigma_k^\varepsilon(X)) = \Sigma_{I'}(X)$ where $I' = \sigma_k^*(I)$.*

(c) *If* k *does not belong to* I *but* $k+1$ *does, then* $\Sigma_I(\sigma_k^\varepsilon(X)) = \Sigma_{I'}(X)$, *where* $I' = \sigma_k^*(I)$.

(d) *If* k, $k+1 \in I$, *then* $\Sigma_I(\sigma_k^\varepsilon(X)) = \Sigma_I(X)$.

The functions Σ_I *satisfy*

$$\Sigma_I(\sigma_k^\varepsilon X) = \Sigma_{I\subset I'}(\sigma_k^\varepsilon)\Sigma_{I'}(X),$$

where $I' = \sigma_k^*(I)$ *and* $X \in F(n)$.

Proof. In each case below we only prove the result for $\varepsilon = +1$, the other case being similar. (a) Suppose that k, $k+1 \in \{1, 2, \dots, n\}\backslash I$. Then $n(k) = n(k+1)$ and so we have

(i) $\Sigma_I(\sigma_k(x_k)) = \Sigma_I(x_{k+1}) = x_{k-n(k)+1} = \sigma_{k-n(k)}(x_{k-n(k)}) = \sigma_{k-n(k)}(\Sigma_I(x_k))$;

(ii) $\Sigma_I(\sigma_k(x_{k+1})) = \Sigma_I(x_{k+1}^{-1}x_kx_{k+1}) = x_{k+1-n(k)}^{-1}x_{k-n(k)}x_{k+1-n(k)} = \sigma_{k-n(k)}(\Sigma_I(x_{k+1}))$; and

(iii) If $j \neq k, k+1$, then $\Sigma_I(\sigma_k(x_j)) = \Sigma_I(x_j) = x_{j-n(j)}$ and $\sigma_{k-n(k)}(\Sigma_I(x_j)) = \sigma_{k-n(k)}(x_{j-n(j)})$.

To see that $x_{j-n(j)} = \sigma_{k-n(k)}(x_{j-n(j)})$ we consider case $1: j < k$, and case $2: j > k+1$. In case 1 we have two subcases; (a) $j \in I$; and (b) j not in I. If (a) then $x_{j-n(j)} = \text{id}$ and we are O.K. If (b), then $n(k)-n(j) \leq k-j-1 < k-j$ showing that $k-n(k) > j-n(j)$, which gives the result. If we have case 2, then again we consider the two cases (a) and (b) above. If we have (a), then $x_{j-n(j)} = \text{id}$ and we are done. If (b), then k, $k+1$ not in I means that $n(j)-n(k) \leq j-k-2$ and so $j-n(j) > k-n(k)+1$, which gives the result. This proves part (a) of Lemma 2.4.

(b) Suppose that $k \in I$, but $k+1 \in \{1, \dots, n\}\backslash I$. Let $I' = \sigma_k^*(I) = (I\backslash\{k\})\cup\{k+1\}$. Then (i) $\Sigma_I\sigma_k(x_k) = \Sigma_I(x_{k+1}) = x_{k+1-n(k+1,I)}$ and $\Sigma_{I'}(x_k) = x_{k-n(k,I')}$; but it is easily seen that $n(k, I') = n(k+1, I)$, and so the result follows in this case. (ii) We have $\Sigma_I\sigma_k(x_{k+1}) = \Sigma_I(x_{k+1}^{-1}x_kx_{k+1}) = \text{id}$ and $\Sigma_{I'}(x_{k+1}) = \text{id}$. (iii) If $j \neq k$, $k+1$, then $\Sigma_I\sigma_k(x_j) = \Sigma_I(x_j) = x_{j-n(j,I)}$ and $\Sigma_{I'}(x_j) = x_{j-n(j,I')}$. The result now follows from the fact that $n(j, I) = n(j, I')$ if $j \neq k$, $k+1$.

(c) Suppose that $k+1 \in I$, but $k \in \{1, \dots, n\}\backslash I$. Let $I' = \sigma_k^*(I) = (I\backslash\{k+1\})\cup\{k\}$. Then (i) $\Sigma_I\sigma_k(x_k) = \Sigma_I(x_{k+1}) = \text{id}$ and $\Sigma_{I'}(x_k) = \text{id}$. (ii) $\Sigma_I\sigma_k(x_{k+1}) = \Sigma_I(x_{k+1}^{-1}x_kx_{k+1}) = x_{k-n(k,I)}$ and $\Sigma_{I'}(x_{k+1}) = x_{k-n(k+1,I')}$; but it is easily seen that $n(k+1, I') = n(k, I)$ and so the result follows in this case. (iii) If $j \neq k$, $k+1$, then $\Sigma_I\sigma_k(x_j) = \Sigma_I(x_j) = x_{j-n(j,I)}$ and $\Sigma_{I'}(x_j) = x_{j-n(j,I')}$. The result now follows since $n(j, I) = n(j, I')$ if $j \neq k$, $k+1$.

(d) If k, $k+1 \in I$, then $\Sigma_I(\sigma_k^\varepsilon(x_j)) = x_{j-n(j)} = \Sigma_I(x_j)$ for all j.

The remaining formula is easily checked using parts (a), (b), (c), and (d), which cover all cases. □

In light of the above result we are now justified in writing $\Sigma_I\sigma_k^\varepsilon\alpha = \Sigma_{I\cup I'}(\sigma_k)^\varepsilon\Sigma_{I'}\alpha$ where $\alpha \in B_n$. We do this in what follows.

The idea for the proof of Proposition 2.3 is to show that Σ_I has the same properties as does φ_I; so we next prove the analogue of Lemma 2.1:

Lemma 2.5. *Let* $\alpha, \beta \in B_n$. *Then* $\Sigma_I(\alpha\beta) = \Sigma_I(\alpha)\Sigma_J(\beta)$, *where* $J = \alpha^*(I)$.

Proof. We induct on the length r of the word α. Clearly the result is true in the case $r = 0$ ($\alpha =$ identity). So now suppose that the result is true for all words α of length $N \geq 0$ and suppose that $\alpha = \sigma_k^\varepsilon \alpha'$, where $\varepsilon = \pm 1$ and the length of α' is N. By induction we now have $\Sigma_I(\alpha'\beta) = \Sigma_I(\alpha')\Sigma_J(\beta)$ for all $I \subset \{1, \dots, n\}$, with $J = (\alpha')^*(I)$. Thus if $I \subset \{1, \dots, n\}$, $I' = (\sigma_k^\varepsilon)^*(I)$ and $J' = (\alpha')^*(I')$, then by Lemma 2.4 we have

$$\begin{aligned}\Sigma_I(\alpha\beta) &= \Sigma_I(\sigma_k^\varepsilon\alpha'\beta) = \Sigma_{I\cup I'}(\sigma_k^\varepsilon)\Sigma_{I'}(\alpha'\beta)\\ &= \Sigma_{I\cup I'}(\sigma_k^\varepsilon)\Sigma_{I'}(\alpha')\Sigma_{J'}(\beta)\\ &= \Sigma_I(\sigma_k^\varepsilon\alpha')\Sigma_{J'}(\beta),\end{aligned}$$

which is what we require since

$$J' = (\alpha')^*(I') = (\alpha')^*((\sigma_k^\varepsilon)^*(I)) = (\sigma_k^\varepsilon\alpha')^*(I). \quad \square$$

Now Proposition 2.3 follows from the last result and the fact that $\varphi_I\sigma_k^\varepsilon = \Sigma_I\sigma_k^\varepsilon$ for all $I \subset \{1, \dots, n\}$, $\varepsilon = \pm 1$, and $k = 1, \dots, n-1$. $\square$

Lemma 2.6. *Let* $\{y_1, \dots, y_m\}$ *be a set of coset representatives for* Y_I *in* B_n. *Let* $J = \{1, 2, \dots, i\}$ *and* $I = \{i+1, i+2, \dots, n\}$. *If* $\gamma \in B_n$ *and* $\alpha = \alpha_1\alpha_2 \in B_n^i$, *where* $\alpha_1 \in \langle\sigma_1, \dots, \sigma_{i-1}\rangle$ *and* $\alpha_2 \in \langle\sigma_{i+1}, \dots, \sigma_{n-1}\rangle$, *then there is* $k = 1, \dots, m$ *such that* $\varphi_I(y_k\gamma\alpha\gamma^{-1}y_k^{-1})$ *is a conjugate of* α_1 *in* B_n *and* $\varphi_J(y_k\gamma\alpha\gamma^{-1}y_k^{-1})$ *is a conjugate of* α_2 *in* B_n.

Proof. For this choice of γ there is $k = 1, \dots, m(i)$ such that $y_k\gamma$ is in Y_I. Then by Lemma 2.1 we see that

$$\begin{aligned}\varphi_I(y_k\gamma\alpha\gamma^{-1}y_k^{-1}) &= \varphi_I(y_k\gamma)\varphi_I(\alpha\gamma^{-1}y_k^{-1})\\ &= \varphi_I(y_k\gamma)\varphi_I(\alpha)\varphi_I(\gamma^{-1}y_k^{-1})\\ &= \varphi_I(y_k\gamma)\varphi_I(\alpha_1\alpha_2)\varphi_I(\gamma y_k)^{-1}\\ &= \varphi_I(y_k\gamma)\alpha_1\varphi_I(\gamma y_k)^{-1}\end{aligned}$$

is a conjugate of α_1 in B_n as required. A similar argument shows that $\varphi_J(y_k\gamma\alpha\gamma^{-1}y_k^{-1})$ is a conjugate of $\varphi_J(\alpha_2)$; but one easily sees that $\varphi_J(\alpha_2)$ is a conjugate of α_2 in B_n and so the result follows. $\square$

For the moment let us keep the notation of Lemma 2.6. Now if $m \geq i + j - 1$, then we let $\Pi_i\colon B_j \to B_m$ be the monomorphism determined by $\Pi_i(\sigma_k) = \sigma_{k+i-1}$ for all $k = 1, \dots, j-1$. Then $\Pi_{i+1}\Sigma_j(\Sigma_h) = \Sigma_h$ for all $h = i+1, \dots, n-1$. This fact, together with the proof of Lemma 2.6, gives the following result.

Corollary 2.7. *If* $\alpha = \alpha_1\alpha_2 \in B_n^i$, *where* $\alpha_1 \in \langle\sigma_1, \dots, \sigma_{i-1}\rangle$ *and* $\alpha_2 \in \langle\sigma_{i+1}, \dots, \sigma_{n-1}\rangle$, *then* α *is conjugate in* B_n *to*

$$(\varphi_I(y_k\gamma\alpha\gamma^{-1}y_k^{-1}))(\Pi_{i+1}\varphi_J(y_k\gamma\alpha\gamma^{-1}y_k^{-1})). \quad \square$$

We now describe the algorithm for determining if we can conjugate $\alpha \in B_n$ into some B_n^i. This is based on the above results and the fact that the conjugacy problem is solved in B_n.

Step 1. For each $i = 1, 2, \dots, n$ we let $I = \{1, 2, \dots, n\}\backslash\{i\}$ and find a set $\{y_1, \dots, y_{m(i)}\}$ of coset representatives for Y_I in B_n.

Do Steps 2(i) and 3(i) for each $i = 1, \dots, m(i)$. If α is conjugate to a braid in B_n^i, then we succeed for some value of i, by Corollary 2.7.

Step 2(i). Let $w_i = (\varphi_I(y_k\gamma\alpha\gamma^{-1}y_k^{-1}))(\Pi_{i+1}\varphi_J(y_k\gamma\alpha\gamma^{-1}y_k^{-1}))$ where we calculate this braid using Proposition 2.3. The braid w_i has a representation as a product of the standard generators and their inverses, which is calculated using an algorithm described in the proof of a theorem of Artin (see [Bi, p. 30]).

Step 3(i). Use the solution of the conjugacy problem as presented in [Ga] (see also [Bi]) to determine whether w_i and α are conjugate in B_n.

If no w_i is conjugate to α, for all $k = 1, \dots, m(i)$, then α is not conjugate to a braid in B_n^i by Corollary 2.7.

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