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ON THE INTEGRABILITY AND L^1-CONVERGENCE OF COMPLEX TRIGONOMETRIC SERIES

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ABSTRACT. We prove that if a weakly even sequence $\{c_k: k = 0, \pm 1, \dots\}$ of complex numbers is such that for some $p > 1$ we have

$$\sum_{m=1}^{\infty} 2^{m/q} \left(\sum_{k=2^{m-1}}^{2^m - 1} |\Delta(c_k + c_{-k})|^p \right)^{1/p} < \infty, \qquad \frac{1}{p} + \frac{1}{q} = 1,$$

then the symmetric partial sums of the trigonometric series $(*)$ $\sum_{k=-\infty}^{\infty} c_k e^{ikx}$ converge pointwise, except possibly at $x = 0 \pmod{2\pi}$, to a Lebesgue integrable function, $(*)$ is the Fourier series of its sum, and series $(*)$ converges in $L^1(-\pi, \pi)$-norm if and only if $\lim_{|k|\to\infty} c_k \ln |k| = 0$.

In addition, we present new proofs of the theorems by J. Fournier and W. Self [6] and by Č. V. Stanojević and V. B. Stanojević [10].

1. INTRODUCTION

Let $\{c_k: k = 0, \pm 1, \dots\}$ be a sequence of complex numbers. We consider the formal trigonometric series

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} \tag{1.1}$$

with symmetric partial sums defined by

$$s_n(x) := \sum_{k=-n}^{n} c_k e^{ikx} \qquad (n = 0, 1, \dots).$$

We assume that $\{c_k\}$ is a null sequence of bounded variation in the sense that

$$\lim_{|k|\to\infty} c_k = 0, \tag{1.2}$$

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(1.3)
$$\sum_{k=-\infty}^{\infty} |\Delta c_k| < \infty,$$

where we adopt the convention that $\Delta c_k := c_k - c_{k+1}$ if $k \geq 0$ and $= c_k - c_{k-1}$ if $k < 0$. In §5 we prove that, under conditions (1.2) and (1.3), series (1.1) converges pointwise, except possibly at $x = 0 \pmod{2\pi}$, to a finite function $f(x)$, say.

2. Results

Let $p > 1$ be a real number. Denote by q the conjugate exponent to p, i.e., $1/p + 1/q = 1$, by I_m the dyadic interval $[2^{m-1}, 2^m)$ for $m \geq 1$, and by $\|\cdot\|$ the $L^1 := L^1(-\pi, \pi)$-norm: $\|f\| := \int_{-\pi}^{\pi} |f(x)|\,dx$.

We prove the following three theorems:

Theorem 1. *If condition* (1.2) *is satisfied and*

(2.1)
$$\sum_{m=1}^{\infty} 2^{m/q} \left(\sum_{|k| \in I_m} |\Delta c_k|^p \right)^{1/p} < \infty \quad \textit{for some } p > 1,$$

then

(i) $f \in L^1$ *if and only if*

(2.2)
$$\sum_{k=1}^{\infty} \frac{|c_k - c_{-k}|}{k} < \infty;$$

(ii) *if* $f \in L^1$, *then* (1.1) *is the Fourier series of* f; *and*

(iii) $\lim_{n\to\infty} \|s_n - f\| = 0$ *if* $\lim_{|n|\to\infty} c_n \ln |n| = 0$.

If $\{c_k\}$ *is an even or odd sequence (see Remarks* 2, 3), *then the conjunctive "if" is replaced by "if and only if."*

We note that (i) was proved by Fournier and Self [6, Corollary 3]. In §5 we give a proof different from theirs.

Problem 1. We are unable to prove the "only if" part in (iii) when $\{c_k\}$ is neither even nor odd. However, we conjecture that it is true in the general case.

Theorem 2. *If condition* (1.2) *is satisfied,*

(2.3)
$$\sum_{k=1}^{\infty} |\Delta(c_k - c_{-k})| \ln k < \infty,$$

(2.4)
$$\sum_{m=1}^{\infty} 2^{m/q} \left(\sum_{k \in I_m} |\Delta(c_k + c_{-k})|^p \right)^{1/p} < \infty \quad \textit{for some } p > 1,$$

then

(i) $f \in L^1$;

(ii) *series* (1.1) *is the Fourier series of* f; *and*

(iii) $\lim_{n\to\infty} \|s_n - f\| = 0$ *if and only if* $\lim_{|n|\to\infty} c_n \ln |n| = 0$.

According to [10], a null sequence $\{c_k\}$ is said to be weakly even if condition (2.3) is satisfied.

Theorem 3. *If conditions* (1.2) *and* (2.3) *are satisfied, and*

$$\mathscr{C}_p := \sum_{m=1}^{\infty} 2^{m/q} \left(\sum_{k \in I_m} |\Delta c_k|^p \right)^{1/p} < \infty \quad \textit{for some } p > 1 , \tag{2.5}$$

then statements (i)–(iii) *of Theorem* 2 *hold.*

By Hölder's inequality, the conditions in Theorems 1–3 imply that $\{c_k\}$ is a null sequence of bounded variation. Thus, the sum $f(x)$ of series (1.1) exists everywhere, except possibly at $x = 0 \pmod{2\pi}$.

Examples show that Theorems 1–3 are not comparable with one another.

Example 1. Let $c_k = (-1)^k (k \ln^2 k)^{-1}$ for $k \geq 2$, and $= 0$ otherwise. Then conditions (2.1) and (2.2) are satisfied, but (2.3) is not. Theorem 1 applies, while Theorems 2, 3 do not.

Example 2. Let $\{c_k\}$ be defined by condition (1.2) and $\Delta c_k = m^{-3}$ for $k = 2^m$, $= -m^{-3}$ for $k = -2^m$ with $m \geq 0$, and $= 0$ otherwise. Conditions (2.3) and (2.4) are satisfied, but conditions (2.2) and (2.5) are not. Thus, only Theorem 2 applies in this case.

Example 3. Let $c_k = k^{-1}$ for $k \geq 1$, $= m^{-3}$ for $k = -2^m$ with $m \geq 0$, and $= 0$ otherwise. Conditions (2.3) and (2.5) are satisfied, but conditions (2.2) and (2.4) are not. Theorem 3 applies, while Theorems 1 and 2 do not.

3. Corollaries and remarks

We draw four corollaries of Theorems 1–3, which are known results.

Remark 1. Theorems 1–3 are stronger when p is closer to 1. For example, by Hölder's inequality, $\mathscr{C}_{p_1} \leq \mathscr{C}_{p_2}$ if $0 < p_1 < p_2$ (see (2.5)). In particular, $\mathscr{C}_1 = \sum |\Delta c_k| \leq \mathscr{C}_p$ if $p > 1$.

Remark 2. If $\{c_k\}$ is an even sequence, i.e., $c_{-k} = c_k$ for $k \geq 1$, then series (1.1) is a cosine series:

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} = 2 \left(\frac{1}{2} c_0 + \sum_{k=1}^{\infty} c_k \cos kx \right) . \tag{3.1}$$

In this case, conditions (2.1) and (2.3) are trivially satisfied and Theorems 1–3 can be reformulated as follows.

Theorem A (Fomin [5]). *If conditions* (1.2) *and* (2.5) *are satisfied, then statements* (i)–(iii) *of Theorem* 2 *hold for the cosine series in* (3.1).

Remark 3. If $\{c_k\}$ is an odd sequence, i.e., $c_{-k} = -c_k$ for $k \geq 1$ and $c_0 = 0$, then series (1.1) is a sine series:

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} = -2i \sum_{k=1}^{\infty} c_k \sin kx. \tag{3.2}$$

In this case, condition (2.4) is automatically satisfied, while conditions (2.1) and (2.3) are of the form

$$\sum_{k=1}^{\infty} |c_k|/k < \infty, \tag{3.3}$$

$$\sum_{k=1}^{\infty} |\Delta c_k| \ln k < \infty, \tag{3.4}$$

respectively, and Theorems 1 and 2 can be reformulated as follows:

Theorem B (Fomin [5]). *If conditions* (1.2), (2.5), *and* (3.3) *are satisfied, then statements* (i)–(iii) *of Theorem* 2 *hold for the sine series in* (3.2).

Theorem C (see, e.g., [1, p. 26, the first part of Theorem 5.1]). *If conditions* (1.2) *and* (3.4) *are satisfied, then statements* (i)–(iii) *of Theorem* 2 *hold for the sine series in* (3.2).

With regard to statement (iii), conditions (1.2) and (3.4) imply that

$$|c_n| \ln n \leq \sum_{k=n}^{\infty} |\Delta c_k| \ln k \to 0,$$

and so in the case of Theorem C, we have $\|s_n - f\| \to 0$ as $n \to \infty$.

Remark 4. According to [1, p. 26], conditions (1.2) and (3.4) imply (3.3). Unfortunately, the converse is not true (as stated incorrectly in [1, p. 26, the second part of Theorem 5.1]). The following is a counterexample:

Example 4. Let $c_k = (\ln m \ln\ln m)^{-1}$ for $k = m^2$ with $m \geq 3$, and $= 0$ otherwise. Then conditions (1.2) and (3.3) are satisfied, even $c_k \ln k \to 0$ as $k \to \infty$, but (3.4) is not satisfied.

However, if $\{c_k\}$ is a nonincreasing null sequence of real numbers, then conditions (3.3) and (3.4) are equivalent.

Remark 5. Following [10], a sequence $\{c_k\}$ of complex numbers is said to belong to the extended Sidon–Telyakovskii class $\mathscr{S}_p^*$ for some $p > 0$ if conditions (1.2) and (2.3) are satisfied, and there exists a nonincreasing sequence $\{A_k\colon k = 1, 2, \ldots\}$ of positive numbers such that

$$\sum_{k=1}^{\infty} A_k < \infty, \tag{3.5}$$

$$\frac{1}{n} \sum_{k=1}^{n} |\Delta c_k|^p / A_k^p = \mathscr{O}(1) \qquad (n = 1, 2, \ldots). \tag{3.6}$$

Now, Theorem 3 implies the following:

Theorem D (Č. V. Stanojević and V. B. Stanojević [10]). *If* $\{c_k\} \in \mathscr{S}_p^*$ *for some* $p > 1$, *then statements* (i)–(iii) *of Theorem* 2 *hold.*

Since condition (3.6) depends on a monotone sequence $\{A_k\}$, which is used as a comparison tool, it is of some interest to replace (3.5) and (3.6) by condition (2.5), hence involving only $\{c_k\}$.

To conclude Theorem D from Theorem 3, we show that conditions (3.5) and (3.6) imply (2.5). In fact, by (3.6),

$$\frac{1}{2^m - 1} \sum_{k \in I_m} |\Delta c_k|^p / A_k^p = \mathscr{O}(1) \qquad (m = 1, 2, \ldots).$$

Since $\{A_k\}$ is monotone,

$$\left(\sum_{k \in I_m} |\Delta c_k|^p \right)^{1/p} \le K 2^{m/p} A_{2^{m-1}}$$

with an absolute constant $K > 0$. Hence,

$$\mathscr{C}_p \le K \sum_{m=1}^{\infty} 2^{m/q + m/p} A_{2^{m-1}} = 2K \sum_{m=0}^{\infty} 2^m A_{2^m}.$$

The last series converges due to (3.5) and the monotone property of $\{A_k\}$. Consequently, condition (2.5) is satisfied and Theorem 3 applies.

4. Sidon type inequalities

Let

$$D_n(x) := \frac{1}{2} + \sum_{k=1}^{n} \cos kx = \frac{\sin(2n+1)x/2}{2 \sin x/2} \qquad (n = 0, 1, \ldots)$$

be the Dirichlet kernel. Then the conjugate expression is

$$\overline{D}_n(x) := -\frac{\cos(2n+1)x/2}{2 \sin x/2}.$$

The latter is connected with the conjugate Dirichlet kernel

$$\widetilde{D}_n(x) := \sum_{k=1}^{n} \sin kx = \frac{\cos x/2 - \cos(2n+1)x/2}{2 \sin x/2} \qquad (n = 1, 2, \ldots)$$

by the identity

$$\widetilde{D}_n(x) = \overline{D}_n(x) - \overline{D}_0(x) \qquad (n = 0, 1, \ldots) \tag{4.1}$$

with the agreement $\widetilde{D}_0(x) = 0$.

The following inequalities play key roles in the proof of Theorems 1–3.

Lemma 1 (Bojanic and Stanojević [2]). *For all* $1 < p \le 2$, *sequences* $\{b_k\colon k = 1, 2, \ldots\}$ *of complex numbers, and integers* $n \ge 1$, *we have*

$$\int_0^\pi \left| \sum_{k=n}^{2n-1} b_k D_k(x) \right| dx \le K_p n^{1/q} \left(\sum_{k=n}^{2n-1} |b_k|^p \right)^{1/p}, \tag{4.2}$$

where K_p *is a constant depending only on* p.

Analogously, one can prove the following (see [11] for a special case):

Lemma 2. *Under the conditions of Lemma* 1, *we have*

$$\int_{\pi/n}^\pi \left| \sum_{k=n}^{2n-1} b_k \overline{D}_k(x) \right| dx \le K_p n^{1/q} \left(\sum_{k=n}^{2n-1} |b_k|^p \right)^{1/p}. \tag{4.3}$$

We are going to use more sophisticated inequalities, which are ultimately consequences of Lemmas 1 and 2. To this effect, we assume that $\{b_k\colon k = 0, 1, \ldots\}$ is a sequence of complex numbers such that $\sum |b_k| < \infty$.

Lemma 3. *For all* $1 < p \le 2$, *we have*

$$\int_0^\pi \left| \sum_{k=0}^\infty b_k D_k(x) \right| dx \le K_p \left\{ |b_0| + \sum_{m=1}^\infty 2^{m/q} \left(\sum_{k \in I_m} |b_k|^p \right)^{1/p} \right\}. \tag{4.4}$$

Lemma 4. *For all* $1 < p \le 2$ *and integers* $s \ge 1$, *we have*

$$\begin{aligned} &\left| \int_{\pi 2^{-s}}^\pi \left| \sum_{k=0}^\infty b_k \overline{D}_k(x) \right| dx - \sum_{j=1}^{2^s - 1} \frac{1}{j} \left| \sum_{k=0}^{j-1} b_k \right| \right| \\ &\qquad \le K_p \left\{ |b_0| + \sum_{m=1}^\infty 2^{m/q} \left(\sum_{k \in I_m} |b_k|^p \right)^{1/p} \right\}. \end{aligned} \tag{4.5}$$

We note that Lemma 3 is a simple corollary of Lemma 1, after grouping the terms in the integrand on the left-hand side of (4.4). Lemma 4 is essentially a consequence of Lemma 2, but the proof of (4.5) is more involved (see [9] for details).

Remark 6. We remind the reader that, under conditions (1.2) and (1.3) (assume this time that $c_k = 0$ for $k < 0$), for all $x \ne 0 \pmod{2\pi}$ we have

$$\frac{1}{2} c_0 + \sum_{k=1}^\infty c_k \cos kx = \sum_{k=0}^\infty D_k(x) \Delta c_k,$$

$$\sum_{k=1}^\infty c_k \sin kx = \sum_{k=1}^\infty \widetilde{D}_k(x) \Delta c_k = \sum_{k=0}^\infty \overline{D}_k(x) \Delta c_k$$

with the agreement $c_0 = 0$ in the second case (cf. (4.1)). Thus, from (4.4) and (4.5) it follows that if conditions (1.2) and (2.5) are satisfied, then for all

$1 < p \le 2$ and integers $s \ge 1$ we have

$$\int_0^\pi \left| \frac{1}{2}c_0 + \sum_{k=1}^\infty c_k \cos kx \right| dx \le K_p(|\Delta c_0| + \mathscr{C}_p),$$

$$\left| \int_{\pi 2^{-s}}^\pi \left| \sum_{k=1}^\infty c_k \sin kx \right| dx - \sum_{k=1}^{2^s-1} |c_k|/k \right| \le K_p \mathscr{C}_p.$$

The last two inequalities were proved by Telyakovskii [11] in the special case when $\{c_k\}$ belongs to the so-called Sidon–Telyakovskii class $\mathscr{S}$. By definition, $\{c_k\colon k = 0, 1, \dots\} \in \mathscr{S}$ if there exists a nonincreasing sequence $\{A_k\colon k = 0, 1, \dots\}$ of nonnegative numbers such that condition (3.5) is satisfied and $|\Delta c_k| \le A_k$ for all k.

5. Proofs of Theorems 1–3

First, we prove the pointwise convergence of series (1.1) under weaker conditions than those imposed in Theorems 1–3.

Lemma 5. *If conditions* (1.2) *and* (1.3) *are satisfied, then series* (1.1) *converges for all* $x \ne 0 \pmod{2\pi}$.

Proof. By summation by parts, we obtain

$$\begin{aligned} s_n(x) &= c_0 + \sum_{k=1}^n (c_k + c_{-k}) \cos kx + i \sum_{k=1}^n (c_k - c_{-k}) \sin kx \\ &= \sum_{k=0}^{n-1} D_k(x) \Delta(c_k + c_{-k}) + (c_n + c_{-n}) D_n(x) \\ &\quad + i \sum_{k=1}^{n-1} \widetilde{D}_k(x) \Delta(c_k - c_{-k}) + i(c_n - c_{-n}) \widetilde{D}_n(x) \\ &= \sum_{k=0}^{n-1} D_k(x) \Delta(c_k + c_{-k}) + i \sum_{k=1}^{n-1} \widetilde{D}_k(x) \Delta(c_k - c_{-k}) \\ &\quad + c_n E_n(x) + c_{-n} E_{-n}(x) - \tfrac{1}{2}(c_n + c_{-n}), \end{aligned} \tag{5.1}$$

where $\Delta(c_k + c_{-k}) = 2c_0 - c_1 - c_{-1}$ for $k = 0$, and

$$E_n(x) = \sum_{k=0}^n e^{ikx} \qquad (n = 0, \pm 1, \dots).$$

By the boundedness of the kernels $D_n(x)$, $\widetilde{D}_n(x)$, and $E_n(x)$ for $x \ne 0 \pmod{2\pi}$ and by (1.3), we conclude that both series,

$$\sum_{k=0}^\infty D_k(x) \Delta(c_k + c_{-k}) \quad \text{and} \quad \sum_{k=1}^\infty \widetilde{D}_k(x) \Delta(c_k - c_{-k}),$$

converge absolutely, and by (1.2) that

$$\lim_{n\to\infty} \{c_n E_n(x) + c_{-n} E_{-n}(x) - \tfrac{1}{2}(c_n + c_{-n})\} = 0.$$

Consequently, series (1.1) converges for all $x \neq 0 \pmod{2\pi}$ and we have

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} = \sum_{k=0}^{\infty} D_k(x)\Delta(c_k + c_{-k}) + i\sum_{k=1}^{\infty} \widetilde{D}_k(x)\Delta(c_k - c_{-k}) \tag{5.2}$$
$$=: f(x), \text{ say.}$$

Proof of Theorem 1. (i) We rewrite (5.2) in the form (see (4.1)):

$$f(x) = \sum_{k=0}^{\infty} D_k(x)\Delta(c_k - c_{-k}) + i\sum_{k=0}^{\infty} \overline{D}_k(x)\Delta(c_k - c_{-k}) \tag{5.3}$$
$$=: f_1(x) + if_2(x), \text{ say,}$$

where $\Delta(c_k - c_{-k}) = c_1 - c_{-1}$ for $k = 0$. By (1.2) and (2.1), Lemma 3 implies $f_1 \in L^1$. Consequently, $f \in L^1$ if and only if $f_2 \in L^1$. According to Lemma 4, $f_2 \in L^1$ if and only if condition (2.2) is satisfied.

(ii) Assume $f \in L^1$. The idea we use is due to Buntinas and Tanović-Miller [4]. By (5.3), we write

$$f(x) = \frac{g(x/2)}{2\sin x/2} \quad \text{for } x \neq 0 \pmod{2\pi},$$

where

$$g(x) := \sum_{k=0}^{\infty}\{\Delta(c_k + c_{-k})\sin(2k+1)x - i\Delta(c_k - c_{-k})\cos(2k+1)x\}. \tag{5.4}$$

Since (2.1) implies (1.3), the series on the right-hand side of (5.4) converges absolutely. So, it is the Fourier series of its sum g and we write

$$\Delta(c_k + c_{-k}) = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin(2k+1)x\,dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\{\cos kx - \cos(k+1)x\}\,dx.$$

From this it follows that

$$2c_0 - (c_n + c_{-n}) = \sum_{k=0}^{n-1}\Delta(c_k + c_{-k}) = \widehat{f}_c(0) - \widehat{f}_c(n), \tag{5.5}$$

where the $\widehat{f}_c(n)$ $(n = 0, 1, \dots)$ are the cosine Fourier coefficients of f. By the Riemann-Lebesgue lemma, $f \in L^1$ implies $\widehat{f}_c(n) \to 0$ as $n \to \infty$. Thus, letting n tend to ∞ in (5.5) yields $\widehat{f}_c(0) = 2c_0$, and consequently,

$$\widehat{f}_c(n) = c_n + c_{-n} \qquad (n = 0, 1, \dots). \tag{5.6}$$

In a similar way, we obtain

$$-i\Delta(c_k - c_{-k}) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\{\sin(k+1)x - \sin kx\}\,dx$$

whence deduce

$$\widehat{f}_s(n) = i(c_n - c_{-n}) \qquad (n = 1, 2, \ldots), \tag{5.7}$$

where the $\widehat{f}_s(n)$ are the sine Fourier coefficients of f.

Now, relations (5.6) and (5.7) yield statement (ii).

(iii) Denote by $\sigma_n(x)$ the first arithmetic mean of series (1.1):

$$\begin{aligned}\sigma_n(x) &= \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) c_k e^{ikx} \qquad (n = 0, 1, \ldots) \\ &= c_0 + \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right)(c_k + c_{-k}) \cos kx \\ &\quad + i \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right)(c_k - c_{-k}) \sin kx.\end{aligned}$$

As is well known, $f \in L^1$ implies $\|\sigma_n - f\| \to 0$ as $n \to \infty$. Since

$$\big|\, \|f - s_n\| - \|s_n - \sigma_n\| \,\big| \le \|f - \sigma_n\| \tag{5.8}$$

statement (iii) is equivalent to the following:

$$\lim_{n\to\infty} \|s_n - \sigma_n\| = 0 \quad \text{if and only if} \quad \lim_{|n|\to\infty} c_n \ln|n| = 0. \tag{5.9}$$

A summation by parts, coupled with simple calculations, leads to the equality

$$\begin{aligned}s_n(x) - \sigma_n(x) &= \frac{1}{n+1} \sum_{k=0}^{n} D_k(x) \Delta(k(c_k + c_{-k})) + (c_{n+1} + c_{-n-1}) D_n(x) \\ &\quad + \frac{i}{n+1} \sum_{k=1}^{k_0-1} \{(c_k - c_{-k}) - (c_{k_0} - c_{-k_0})\} k \sin kx \\ &\quad - \frac{i}{n+1} \left\{ \sum_{k=k_0}^{n} D_k'(x) \Delta(c_k - c_{-k}) + (c_{n+1} - c_{-n-1}) D_n'(x) \right\},\end{aligned}$$

where "prime" means differentiation with respect to x and $k_0 = 2^{m_0 - 1}$ is an integer fixed later on. Hence it follows that

$$\begin{aligned}\sum := &\left| \|s_n - \sigma_n\| - \left\|(c_{n+1} + c_{-n-1}) D_n - \frac{i}{n+1}(c_{n+1} - c_{-n-1}) D_n'\right\| \right| \\ &\le \frac{1}{n+1} \left\| \sum_{k=0}^{n} D_k \Delta(k(c_k + c_{-k})) \right\| + \frac{2\pi}{n+1} \sum_{k=1}^{k_0-1} k |c_k - c_{-k} - c_{k_0} + c_{-k_0}| \\ &\quad + \frac{1}{n+1} \left\| \sum_{k=k_0}^{n} D_k' \Delta(c_k - c_{-k}) \right\|.\end{aligned} \tag{5.10}$$

Applying Bernstein's inequality [12, Vol. 2, p. 11] and Lemmas 3, 4 gives

$$\begin{aligned}\sum &\le \frac{1}{n+1}\left\{|2c_0 - c_1 - c_{-1}| + \sum_{m=1}^{j} 2^{m/q}\left(\sum_{k\in I_m} |\Delta(k(c_k + c_{-k}))|^p\right)^{1/p}\right\}\\ &+ \frac{2\pi}{n+1}\sum_{k=1}^{k_0-1} k|c_k - c_{-k} - c_{k_0} + c_{-k_0}|\\ &+ \sum_{m=m_0}^{j} 2^{m/q}\left(\sum_{k\in I_m} |\Delta(c_k - c_{-k})|^p\right)^{1/p}\\ &=: \Sigma_1 + \Sigma_2 + \Sigma_3, \text{ say,}\end{aligned}$$

where $j = j(n)$ is the integer for which $2^{j-1} \le n < 2^j$. Given any $\varepsilon > 0$, by (2.1) we choose m_0 so large that $\Sigma_3 < \varepsilon$. Then setting $k_0 = 2^{m_0-1}$, we take n so large that $\Sigma_2 < \varepsilon$. Taking into account conditions (1.2) and (2.1) and the relation

$$\Delta(k(c_k + c_{-k})) = k\Delta(c_k + c_{-k}) - (c_{k+1} + c_{-k-1}),$$

it is not difficult to see that $\Sigma_1 < \varepsilon$ for large enough n. To sum up, we have $\Sigma < 3\varepsilon$ in (5.10) if n is sufficiently large. This means that

$$\text{(5.11)}\quad \lim_{n\to\infty}\left\{\|s_n - \sigma_n\| - \|(c_{n+1} + c_{-n-1})D_n - \frac{i}{n+1}(c_{n+1} - c_{-n-1})D_n'\|\right\} = 0.$$

To complete the proof of statement (iii) on the basis of (5.8)–(5.11), it remains to refer to the facts that both $\|D_n\|$ and $\|D_n'\|/(n+1)$ have the order of magnitude $\ln n$ as $n \to \infty$ (see, e.g., [12, Vol. 1, p. 67] and [8, Lemma 9], respectively).

Proof of Theorem 2. (i) Following an idea of Garrett and Stanojević [7], we introduce

$$\text{(5.12)}\qquad u_n(x) := s_n(x) - c_nE_n(x) - c_{-n}E_{-n}(x) \qquad (n = 0, 1, \ldots)$$

called modified trigonometric sums. By (5.1) we write

$$u_n(x) = \sum_{k=0}^{n-1} D_k(x)\Delta(c_k + c_{-k}) + i\sum_{k=1}^{n-1}\widetilde{D}_k(x)\Delta(c_k - c_{-k}) - \tfrac{1}{2}(c_n + c_{-n}).$$

This and (5.2) imply that

$$\|f - u_n\| \le \left\|\sum_{k=n}^{\infty} D_k\Delta(c_k + c_{-k})\right\| + \sum_{k=n}^{\infty}\|\widetilde{D}_n\|\,|\Delta(c_k - c_{-k})| + \pi|c_n + c_{-n}|.$$

As is known (see, e.g., [12, Vol. 1, p. 67]), $\|\widetilde{D}_k\|$ has the order of magnitude $\ln k$ as $k \to \infty$. By Remark 1 we may assume that $1 < p \le 2$ and apply

Lemma 3. As a result, we obtain that

$$\|f - u_n\| \le K_p \sum_{m=j}^{\infty} 2^{m/q} \left(\sum_{k\in I_m} |\Delta(c_k + c_{-k})|^p \right)^{1/p} + K \sum_{k=n}^{\infty} |\Delta(c_k - c_{-k})| \ln k + \pi |c_n + c_{-n}|,$$

where $j = j(n)$ is again the integer for which $2^{j-1} \le n < 2^j$. Taking (1.2), (2.3), and (2.4) into account yields

$$\lim_{n\to\infty} \|f - u_n\| = 0. \tag{5.13}$$

Since u_n is a polynomial, it follows that f is integrable.

(ii) It is common place that convergence in L^1-norm (so-called strong convergence) implies weak convergence. Let $l \ge 0$ be a fixed integer. By (1.2), (5.12), and (5.13),

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ilx}\,dx = \lim_{n\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} u_n(x)e^{-ilx}\,dx = \lim_{n\to\infty}(c_l - c_n) = c_l$$

and similarly when $l < 0$. This shows that (1.1) is the Fourier series of f.

(iii) By (5.12) and (5.13),

$$\big|\, \|f - s_n\| - \|c_nE_n + c_{-n}E_{-n}\| \,\big| \le \|f - u_n\| \to 0 \qquad (n \to \infty).$$

Since by [3, Lemma 1.1],

$$\lim_{n\to\infty} \|c_nE_n + c_{-n}E_{-n}\| = 0 \quad \text{if and only if} \quad \lim_{|n|\to\infty} c_n \ln|n| = 0,$$

the proof of statement (iii) and Theorem 2 is complete.

Proof of Theorem 3. Instead of (5.1), we start with the representation (see [10, p. 680])

$$\begin{aligned} s_n(x) &= c_0 + \sum_{k=1}^{n} c_k(e^{ikx} + e^{-ikx}) + \sum_{k=1}^{n} (c_{-k} - c_k)e^{-ikx} \\ &= 2\sum_{k=0}^{n-1} D_k(x)\Delta c_k + 2c_nD_n(x) \\ &\quad + \sum_{k=1}^{n-1} (E_{-k}(x) - 1)\Delta(c_{-k} - c_k) + (c_{-n} - c_n)(E_{-n}(x) - 1) \\ &= 2\sum_{k=0}^{n-1} D_k(x)\Delta c_k + \sum_{k=1}^{n-1} (E_{-k}(x) - 1)\Delta(c_{-k} - c_k) \\ &\quad + c_nE_n(x) + c_{-n}E_{-n}(x) - c_{-n}. \end{aligned}$$

The rest are similar to the proof of Theorem 2, and therefore, we omit them.

Problem 2. Similarly to the proofs of Theorems 2, 3, it would be desirable to use the modified trigonometric sums $u_n(x)$ defined in (5.12) or other appropriate sums in order to shorten the proof of Theorem 1. We are unable to carry it out.

Remark 7 *added in proof*. A result of Chen [13] answers our Problem 1 in the affirmative: Under the conditions of Theorem 1, statement (iii) can be replaced by the following stronger one:

(iii$'$) $\lim_{n\to\infty} \|s_n - f\| = 0$ if and only if $\lim_{|n|\to\infty} c_n \ln|n| = 0$.

In fact, by [13, Corollary 3.1], this equivalence relation holds under the conditions that $f \in L^1$, (1.1) is the Fourier series of f, and for some $1 < p \le 2$

$$(*) \qquad \lim_{\lambda\downarrow 1} \limsup_{n\to\infty} \sum_{|k|=n}^{[\lambda n]} |k|^{p-1} |\Delta c_k|^p = 0.$$

Now, it is obvious that condition $(*)$ follows from our condition (2.1). Thus, (iii$'$) is proved.

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