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A FEW REMARKS ON RIESZ SUMMABILITY OF ORTHOGONAL SERIES

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ABSTRACT. We study convergence behavior of some sequences and series related to a given orthogonal series. Following the developed technique we define (in terms of fourth mixed moments only) a class of orthonormal functions $\{X_i\}_{i\geq 1}$ such that the condition: $\exists k \in \mathbb{N}\ \sum_{i\geq 1} \mu_i^2(\ln^{(k)} i)^2 < \infty$ implies almost everywhere convergence of the series $\sum_{i\geq 1} \mu_i X_i$, here for every $i = 1, 2, \dots, j = 1, \dots, k$,

$$\ln^{(1)} i = \ln_2 i, \qquad \ln^{(j)} i = \ln_2(\max(1, \ln^{(j-1)} i)).$$

INTRODUCTION

As is well known, conditions $\sum_{i\geq 1} \mu_i^2(\ln_2 i)^2 = \infty$, $\sum_{i\geq 1} \mu_i^2 < \infty$ imply that there exists an orthonormal system $\{X_i\}_{i\geq 1}$ such that the series $\sum_{i\geq 1} \mu_i X_i$ diverges on the set of positive measure. Hence the only way to assure almost everywhere convergence of the series $\sum_{i\geq 1} \mu_i X_i$, and weaken the condition $\sum_{i\geq 1} \mu_i^2(\ln_2 i)^2 < \infty$, is to impose additional conditions on the system $\{X_i\}_{i\geq 1}$.

There are known systems (e.g. independent, Haar, or trigonometric) for which the condition $\sum_{i\geq 1} \mu_i^2 < \infty$ is equivalent to almost everywhere convergence of the series $\sum_{i\geq 1} \mu_i X_i$.

We define a class of orthonormal systems $\{X_i\}_{i\geq 1}$ that a.e. convergence of the series $\sum_{i\geq 1} \mu_i X_i$ takes place under much weaker condition than $\sum_{i\geq 1} \mu_i^2(\ln_2 i)^2 < \infty$.

More precisely our result is as follows: We use probabilistic terminology throughout the paper. Let $(\Omega, \mathcal{F}, P)$ be the underlying probability space. ($E(\cdot)$ denotes integral with respect to P, a.s. means almost surely that is almost everywhere with respect to P, functions are called random variables and so on.) Denote $\mathbb{L}_2 = L_2(\Omega, \mathcal{F}, P)$. Let $\{X_i\}_{i\geq 1}$, $X_i \in \mathbb{L}_2$ be a se-

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quence of standardized uncorrelated random variables. Let us denote $H_{n,m} = \text{span}(X_{n+1}, \dots, X_m)$, $0 \le n < m \le \infty$, and suppose that the system $\{X_i\}_{i\ge 1}$ satisfies the following two conditions:

(S) $\exists C > 0\ \forall n \in \mathbb{N},\ U \in H_{0,n},\ Z \in H_{n,\infty}$: $EU^2Z^2 \le CEU^2EZ^2$;

(O) $\forall n > k \ge 1\ Z \in H_{0,k},\ T \in H_{k,n},\ U \in H_{0,n},\ W \in H_{n,\infty},\ EW^2 < \infty$: $EZTUW = 0$.

Remark 1. It is easy to notice that if only system $\{X_i\}_{i\ge 1}$ satisfies:

(a) $\forall i, j, k, l \in \mathbb{N},\ i \ne j \ne k \ne i,\ EX_iX_jX_kX_l = 0$,

(b) $\exists C\ \forall i \ne j,\ EX_i^2X_j^2 \le CEX_i^2EX_j^2$,

then it satisfies conditions (S) and (O).

An orthonormal system satisfying (S) and (O) is called pseudo-square orthogonal (PSO system); (it is not exactly square orthogonal). Our main result follows:

Theorem 1. *Let $\{X_i\}_{i\ge 1}$ be any PSO system. The orthogonal series $\sum_{i\ge 1} \mu_i X_i$ converges a.s. if only $\exists k \in \mathbb{N}$ $\sum_{i\ge 1} \mu_i^2 (\ln^{(k)} i)^2 < \infty$. Numbers $\ln^{(j)} i$, $i, j = 1, 2, \dots$ are defined above.*

Remark 2. Notice that (i) every system of independent, standardized random variables is PSO; (ii) a system of standardized, belonging to L_∞, martingale differences is PSO; (iii) one needs not to assume that $E|X_i|^m$ is finite for every $i \ge 1$ and $m > 2$; (iv) condition (a) of Remark 1 implies that the system satisfies condition (O) while both conditions (a) and (b) together imply conditions (S) and (O). On the other hand, condition (S) is obviously less restrictive.

It has to be underlined that the proof of Theorem 1 is relatively simple and is based entirely on the Rademacher–Menshov theorem and on the refined tricks from papers [4, 5]. The properties of Riesz summation procedure were developed in those papers (although the terminology and the purpose of those papers were different).

This paper consists of three parts. In §I we present a collection of properties of Riesz summation procedure and based on them, properties of orthogonal series (without assuming that the underlying system of orthonormal functions is PSO).

In §II we prove further properties of Riesz summation procedure and its application to the orthogonal series under additional assumption that the orthonormal systems considered are PSO. We also prove Theorem 1.

In §III we present applications of Theorem 1 to the laws of large numbers, possible extensions, concluding remarks, and open questions.

I. Auxiliary properties of Riesz summation procedure

Let $\{X_i\}_{i\ge 1}$ be a sequence of standardized, uncorrelated random variables (sequence of SURV) and let $\sum_{i\ge 0} \mu_i X_{i+1}$ [1] be the orthogonal series under con-

[1] For convenience and simplicity, we use the same notation as in [4]. It does not change convergence properties of the considered series in any way.

sideration. Assume that $\sum_{i\geq 0}\mu_i^2 < \infty$. Let us define the following sequences of reals and auxiliary random variables.

Let $\{\nu_i\}_{i\geq 0}$ be a sequence of reals satisfying the following conditions:

(N) $\nu_0 = 1$, $0 \leq \nu_i < 1$, $i \geq 1$, $\sum_{i\geq 0}\nu_i = \infty$.

Let $\{b_i\}_{i\geq 0}$ be another sequence of reals related to the sequence $\{\nu_i\}$ in the following way:

$$b_0 = 1, \qquad \nu_i = b_i \Big/ \sum_0^i b_j, \qquad i \geq 1.$$

In this case we write $\overline{\{\nu_i\}} = \{b_i\}$. Let us denote

$$T_0 = 0, \qquad T_i = \sum_{j=0}^{i-1}(b_j\mu_j X_{j+1}/\nu_j) \Big/ \sum_{j=0}^{i-1} b_j, \qquad i \geq 1;$$

$$S_0 = 0, \qquad S_i = \sum_{j=0}^{i-1}\mu_j X_{j+1}, \qquad i \geq 1;$$

$$\overline{S}_i = \sum_{j=0}^{i} b_j S_j \Big/ \sum_{j=0}^{i} b_j, \qquad i \geq 0.$$

Further let $\{n_k\}_{k\geq 1}$ be a sequence of indexes defined by the following relationship:

$$\sum_{j=n_k+1}^{n_{k+1}} \nu_j = O(1)^2 \qquad k \geq 0. \tag{1}$$

We have the following:

Lemma 1.

(1) $\overline{S}_n = \sum_{i=0}^n \nu_i T_i$ *a.s.*

(2) $\sum_{i\geq 0}\nu_i T_i^2 < \infty$ *a.s.*, $\sum_{i\geq 0}\nu_i E T_i^2 < \infty$.

(3) $V_n \overset{\text{df}}{=} \sum_{i=0}^n b_i S_i^2 / \sum_{i=0}^n b_i - (\overline{S}_n)^2 \xrightarrow[n\to\infty]{} 0$ *a.s.*, $\sum_{i\geq 1}\nu_i V_i < \infty$ *a.s.*

(4) $\sum_{i=0}^n b_i T_i^2 / \sum_{i=0}^n b_i \xrightarrow[n\to\infty]{} 0$ *a.s.*

(5) $T_{n_k} \xrightarrow[k\to\infty]{} 0$ *a.s.*

(6) *Sequence* $\{S_{n_k}\}_{k\geq 0}$ *converges a.s. to some random variable* $S \in \mathbb{L}_2$ *iff the series* $\sum_{i\geq 0}\nu_i T_i$ *converges a.s. to* S.

(7) $T_n \xrightarrow[n\to\infty]{} 0$ *a.s. iff the series* $\sum_{i\geq 0}\mu_i T_i X_{i+1}$ *converges a.s.*

(8) *If* $T_n \xrightarrow[n\to\infty]{} 0$ *a.s. then* ($S_n \xrightarrow[n\to\infty]{} S$ *a.s. iff* $S_{n_k} \xrightarrow[k\to\infty]{} S$) *where* $S \in \mathbb{L}_2$.

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[2] $a_k = \mathrm{O}(1) \Leftrightarrow 0 < \liminf |a_k| \leq \limsup |a_k| < \infty$, $a_k = o(1) \Leftrightarrow \limsup |a_k| = 0$.

Remark 3. Note that assertions (5) and (6) remain true if the sequence $\{n_k\}$ is defined by $1/\sum_{n_k+1}^{n_{k+1}} \nu_i = o(1)$ while assertion (8) is also true if this sequence is defined by $\sum_{n_k+1}^{n_{k+1}} \nu_i = o(1)$.

Remark 4. Assertion (6) together with assertion (1), constitutes a reformulation and correction of Zygmund's theorem [1, Theorem 2.8.7]. As an illustration, let us denote: $\lambda_n = \sum_{i=0}^{n} b_i$, $n \geq 0$. Then we have (compare [4, proof of Lemma 1]) $\lambda_n = \prod_1^n (1-\nu_i)^{-1} \geq \exp(\sum_{i=0}^{n} \nu_i)$. Hence $\lambda_{n_{k+1}}/\lambda_{n_k} \geq \exp(\sum_{n_k+1}^{n_{k+1}} \nu_i) > 1$. On the other hand, notice that we do not need the condition that $\lambda_{n_{k+1}}/\lambda_{n_k} \leq r < \infty$, $k \geq 1$. It was used in [1] to prove that $\sup_{n_k+1 \leq i \leq n_{k+1}} |\overline{S}_i - \overline{S}_{n_k}| \to 0$ a.s. In fact as it follows from estimation (3) only $\sum_{n_k+1}^{n_{k+1}} \nu_i = O(1)$, $k \geq 1$ is necessary.

Remark 5. Note that due to assertion (1) we have the following observation: $\{S_n\}_{n\geq 0}$ is absolutely Riesz summable with respect to the sequence $\{\nu_i\}$ (or using more common notation $|R, \sum_{i=0}^{n} b_i, 1|$) iff the series $\sum_{i\geq 0} \nu_i |T_i|$ converges a.s. Thus, for example, a great part of [3] could be substantially simplified.

Remark 6. Notice that assertion (3) together with $\nu_i = 1/(i+1)$, $i \geq 0$ ($b_i = 1$, $i \geq 0$) is equivalent to Hardy–Littlewood theorem about the equivalence of ordinary and strong Cesaro summability of orthogonal series. Also compare Zygmund's theorem [1, Theorem 2.6.2].

Remark 7. From assertion (4) it follows that $\sum_0^n b_i T_i / \sum_0^n b_i \xrightarrow[n\to\infty]{} 0$ a.s. and $\sum_0^n b_i \nu_i T_i / \sum_0^n b_i \xrightarrow[n\to\infty]{} 0$ a.s., and consequently that ($\overline{S}_n \xrightarrow[n\to\infty]{} S$ a.s. iff $\sum_0^n b_i \overline{S}_i / \sum_0^n b_i \xrightarrow[n\to\infty]{} S$ a.s.).

Proof of Lemma 1. Assertions (1), (3), (4), and (7) almost immediately follow the following:

Numerical Lemma. *Let* $\{\nu_i\}_{i\geq 0}$ *be a sequence of reals satisfying condition* (N). *Let* $\{a_i\} = \overline{\{\nu_i\}}$, *while letting* $\{y_i\}_{i\geq 1}$ *be any sequence of reals. Further let us denote*

$$\overline{y}_n = \sum_{i=0}^{n-1} a_i y_{i+1} \Big/ \sum_{i=0}^{n-1} a_i, \qquad n \geq 1; \qquad s_n = \sum_{i=0}^{n-1} \nu_i y_{i+1}, \qquad n \geq 1;$$

$$\widehat{s}_n = \sum_{i=0}^{n} \nu_i \overline{y}_i, \qquad n \geq 1; \qquad \overline{s}_n = \sum_{i=0}^{n} a_i s_i \Big/ \sum_{i=0}^{n} a_i \quad \textit{with } \overline{y}_0 = s_0 = \overline{s}_0 = \widehat{s}_0 = 0.$$

Then

(a) $\widehat{s}_n = \overline{s}_n$, $n \geq 0$.

(b) *Suppose additionally that* $\sum_{i\geq 0} \nu_i^2 y_{i+1}^2 < \infty$, $\sum_{i\geq 0} \nu_i (\overline{y}_i)^2 < \infty$, *then*

(ba) $v_n \overset{\mathrm{df}}{=} \sum_{i=0}^{n} a_i (s_i - \overline{s}_n)^2 / \sum_{i=0}^{n} a_i = \sum_{i=0}^{n} a_i s_i^2 / \sum_{i=0}^{n} a_i - (\overline{s}_n)^2 \xrightarrow[n\to\infty]{} 0$.

Moreover, we have $\sum_{i\geq 0}\nu_i v_i < \infty$,

(bb) $\sum_0^n a_i(\overline{y}_i)^2 / \sum_0^n a_i \xrightarrow[n\to\infty]{} 0$; *and*

(bc) $\overline{y}_n \xrightarrow[n\to\infty]{} 0$ *iff the series* $\sum_{i\geq 0}\nu_i\overline{y}_i y_{i+1}$ *converges.*

Proof of the numerical lemma. The proof of this lemma is elementary. It is based on the recursive equations satisfied by the sequences $\{\overline{y}_i\}$, $\{\overline{s}_i\}$, and $\{v_n\}$ and on the applications of Theorem 4 of [4].

In order to prove assertion (2) of Lemma 1 let us notice that

$$T_{n+1} = (1-\nu_n)T_n + \mu_n X_{n+1}, \qquad n \geq 1. \tag{2}$$

Now we take squares of both sides, then the expectation and finally we apply Theorem 4 of [4].

To prove assertion (5) let us notice that the sequence $\{T_{n_k}\}$ can be presented in the following recursive form:

$$T_{n_{k+1}} = (1-\vartheta_k)T_{n_k} + \vartheta_k W_k,$$

$$\text{where } 1-\vartheta_k = \sum_{i=0}^{n_k} b_i \Big/ \sum_{i=0}^{n_{k+1}} b_i \text{ and } W_k = \sum_{i=n_k+1}^{n_{k+1}} b_i(\mu_i X_{i+1}/\nu_i) \Big/ \sum_{i=n_k+1}^{n_{k+1}} b_i.$$

Since quadratic function is convex we get

$$T_{n_{k+1}}^2 \leq (1-\vartheta_k)T_{n_k}^2 + \vartheta_k W_k^2.$$

Thus according to Theorem 3″ of [4] $T_{n_k} \xrightarrow[k\to\infty]{} 0$ a.s. if only $\sum_{k\geq 0}\vartheta_k E W_k^2 < \infty$. An easy calculation together with the definitions of the sequences $\{n_k\}$ and $\{b_i\}$, proves that this condition is satisfied.

Assertion (6) follows the identity $S_{n_k} - T_{n_k} = \sum_{i=0}^{n_k-1}\nu_i T_i$ and the following estimation:

$$\sup_{n_k+1\leq i\leq n_{k+1}} \left|\sum_{j=n_k+1}^{i}\nu_j T_j\right|^2 \leq \left(\sum_{j=n_k+1}^{n_{k+1}}\nu_j\right)\left(\sum_{j=n_k+1}^{n_{k+1}}\nu_j T_j^2\right) \xrightarrow[k\to\infty]{} 0 \quad \text{a.s.} \tag{3}$$

Here we used assertion (2) and the definition of the sequence $\{n_k\}$.

Assertion (8) follows (6) and the identity $S_{n+1} = T_{n+1} + \overline{S}_n$. □

Corollary 1.1. *If* $|X_i| \leq M$ *a.s.* $i \geq 1$ *for some real* M *then* $S_n \xrightarrow[n\to\infty]{} S$ *a.s. iff* $S_{n_k} \xrightarrow[k\to\infty]{} 0$ *a.s. for some* $S \in \mathbb{L}_2$. *Sequence* $\{n_k\}$ *is defined by* $\sum_{j=n_k+1}^{n_{k+1}}|\mu_j| = O(1)$, $k \geq 1$.

For the proof apply Lemma 1 with $\nu_i = |\mu_i|/(1+\sup_{i\geq 1}|\mu_i|)$, $i \geq 1$. More precisely first apply assertions (7) and (5) to prove that $T_n \xrightarrow[n\to\infty]{} 0$ a.s. and then apply assertion (8).

II. Further properties of the orthogonal series under assumptions (S) and (O)

The following two properties of S- and PSO-systems are true.

Lemma 2. *Suppose the sytem $\{X_i\}_{i\geq 1}$ satisfies condition (S). Then $\forall k \in \mathbb{N}$ $S_n \xrightarrow[n\to\infty]{} S$ a.s. iff $S_{(n^k)} \xrightarrow[n\to\infty]{} S$ a.s. for some $S \in \mathbb{L}_2$.*

Lemma 3. *Let $\{X_i\}_{i\geq 1}$ be a PSO system and suppose that $\sup_{i\geq 1}|\mu_i|\ln i < \infty$. Then $\forall k \in \mathbb{N}$ $S_n \xrightarrow[n\to\infty]{} S$ a.s. iff $S_{2^{(k)}(n)} \xrightarrow[n\to\infty]{} S$ a.s., where $2^{(1)}(n) = 2^n$, $n \geq 0$, $2^{(k)}(n) = 2^{2^{(k-1)}(n)}$, $k \geq 2, n \geq 0$.*

Since not only assertions but also some ideas of the proofs of these lemmas are used in the proof of Theorem 1, we present proofs of the lemmas first.

The ideas of the proofs of these lemmas are very similar. First notice that if $\sum \mu_i^2(\ln i)^2 < \infty$ then by the Rademacher–Menshov theorem these lemmas are true. Thus we consider the opposite case.

The first step of the proofs of Lemmas 2, 3 is to use assertion (7) of Lemma 1 applied with:

$$\nu_i = |\mu_i| \Big/ \Big(1 + \sup_{i\geq 1}|\mu_i|\Big) \quad \text{in case of Lemma 2;}$$

$$\nu_i = \mu_i^2 \ln^2 i \Big/ \Big(1 + \sup_{i\geq 1}\mu_i^2 \ln^2 i\Big) \quad \text{in case of Lemma 3;}$$

and prove that $T_n \xrightarrow[n\to\infty]{} 0$ a.s.

In the case of Lemma 3 it is exceptionally simple, since we apply the Rademacher–Menshov theorem to the series $\sum_{i\geq 0}\mu_i T_i X_{i+1}$. This can be done since this series is orthogonal ($\{X_i\}_{i\geq 1}$ is a PSO-system). Hence to prove that $T_n \xrightarrow[n\to\infty]{} 0$ a.s. it is enough to show that $\sum_{i\geq 1}\mu_i^2 \ln^2 i ET_i^2 < \infty$.

We have, however, $\sum_{i\geq 1}\mu_i^2 \ln^2 i ET_i^2 = O(1)\sum_{i\geq 1}\nu_i ET_i^2 < \infty$ by assertion (2) of Lemma 1.

To prove that $T_n \xrightarrow[n\to\infty]{} 0$ a.s. under assumptions of Lemma 2, let us consider the following sequence:

$$Z_n = \sum_{i=0}^{n-1} b_i(\mu_i T_i X_{i+1}/\nu_i) \Big/ \sum_{i=0}^{n-1} b_i, \qquad n \geq 1, \tag{4}$$

where $\{b_i\} = \overline{\{\nu_i\}}$.

We consider a recursive form of the sequences $\{Z_n\}$ and $\{T_n^2 - 2Z_n\}$ to deduce by Theorem 4 of [4] that $2Z_n - T_n^2 \xrightarrow[n\to\infty]{} 0$ a.s. We have further:

$$Z_{n+1}^2 \leq (1-\nu_n)Z_n^2 + \nu_n(\mu_n^2 T_n^2 X_{n+1}^2/\nu_n^2).$$

Recalling the definition of the sequence $\{\nu_i\}$ we see that $Z_n \xrightarrow[n\to\infty]{} 0$ a.s. (and $T_n \xrightarrow[n\to\infty]{} 0$ a.s. of course) if only

$$\sum_{n\geq 1} \mu_n T_n^2 X_{n+1}^2 < \infty \quad \text{a.s.} \tag{5}$$

System $\{X_i\}$ however, satisfies assumption (S), besides we have assertion (2) of Lemma 1. Thus we see that (5) is true. The next step in the proofs of Lemmas 2, 3 is to use assertion (8) of Lemma 1 and deduce that $S_n \xrightarrow[n\to\infty]{} S$ a.s. iff $S_{n_k} \xrightarrow[k\to\infty]{} S$ a.s.

Let us analyze the sequence $\{n_k\}$ in the two considered cases. First notice that in the case of Lemma 2 the sequence $\{n_k\}$ is defined by the relationship:

$$\sum_{j=n_k+1}^{n_{k+1}} |\mu_j| = O(1).$$

Hence we have:

$$O(1)k = \sum_{i=0}^{n_k} |\mu_i| \leq \sqrt{n_k}\sqrt{\sum_{i=0}^{n_k} \mu_i^2}.$$

Thus $O(1)k^2 \leq n_k$, $k \geq 1$ since $\sum_{i\geq 0} \mu_i^2 < \infty$.

In case of Lemma 3 the sequence $\{n_k\}$ is given by the relationship:

$$\sum_{j=n_k+1}^{n_{k+1}} \mu_j^2 \ln^2 j = O(1).$$

Then we have

$$\ln^2 n_k \sum_{j=n_k+1}^{n_{k+1}} \mu_j^2 \leq O(1) = \sum_{j=n_k+1}^{n_{k+1}} \mu_j^2 \ln^2 j \leq \ln^2 n_{k+1} \sum_{j=n_k+1}^{n_{k+1}} \mu_j^2.$$

Hence $\sum_{k\geq 1} 1/\ln^2 n_k < \infty$ or equivalently $1/\ln^2 n_k = o(1/k)$ since sequence $\{n_k\}$ is increasing. Thus $n_k \geq o(1)[2^{\sqrt{k}}]$.

Now notice that we can define new orthogonal series $\sum_{k\geq 0} \mu'_k X'_{k+1}$, where

$$(\mu'_k)^2 = \sum_{j=n_k+1}^{n_{k+1}} \mu_j^2, \qquad X'_{k+1} = (1/\mu'_k) \sum_{j=n_k+1}^{n_{k+1}} \mu_j X_{j+1}.$$

Moreover if the orthogonal system $\{X_i\}$ satisfies condition (S) then the system $\{X'_i\}$ also satisfies condition (S). Similarly if the system $\{X_i\}$ was PSO then the system $\{X'_i\}$ is also PSO. Thus we can apply previous considerations to the series $\sum_{k\geq 0} \mu'_k X'_{k+1}$ and deduce that $S_n \xrightarrow[n\to\infty]{} S$ a.s. iff $S_{n_{n_k}} \xrightarrow[k\to\infty]{} S$ a.s.

In particular $S_n \xrightarrow[n\to\infty]{} S$ a.s. iff $S_{(n^2)^2} \xrightarrow[n\to\infty]{} S$ a.s. under assumptions of Lemma 2, and $S_n \xrightarrow[n\to\infty]{} S$ a.s. iff $S_{n^2} \xrightarrow[n\to\infty]{} S$ a.s. iff $S_{2^n} \xrightarrow[n\to\infty]{} S$

a.s. ($\sqrt{n^2} = n$) under assumptions of Lemma 3 for some $S \in \mathbb{L}_2$. Now we can redefine orthogonal series once more and repeat finite number of times arguments already used. This concludes the proofs of Lemmas 2, 3. □

Proof of Theorem 1. If $\sum_{i\geq 1}\mu_i^2(\ln_2 i)^2 < \infty$, then by the Rademacher–Menshov theorem there is nothing to prove. Thus let us consider the opposite case. (Here and below $\ln i = \ln_2 i$, $\ln\ln i = \ln_2(\ln_2 i)$ and so on.) First notice that because of our assumption, there exists a sequence of indexes of the form $\{2^{(k)}(n)\}$ $n \geq 0$ such that the sequence $\{S_{2^{(k)}(n)}\}$ $n \geq 0$ is convergent a.s. Thus if only $\sup_{i\geq 1}|\mu_i|\ln i < \infty$ we can apply Lemma 3 and immediately get the desired assertion.

Hence let us consider the opposite case. Again it follows from the assumptions that the given orthogonal series can be split into a finite sum of orthogonal series in the following way:

$$\sum_{j=1}^{k}\sum_{i\in A_j^c}\mu_i X_{i+1} \stackrel{\text{df}}{=} \sum_{j=1}^{k}\mathfrak{H}_j\,,$$

where

$$A_j^c = \{i\colon\, c/\ln^{(j-1)} i \leq |\mu_i| \leq c/\ln^{(j)} i\}, \qquad c \in \mathbb{R}^+ \ (\ln_j^{(0)} \equiv \infty;\ i > 0).$$

We prove convergence of each of the series $\mathfrak{H}_j$ $j = 1, \dots, k$ separately.

Notice that apart from $\mathfrak{H}_1$, all others are lacunary series. $\mathfrak{H}_1$ is a.s. convergent by Lemma 3. We prove convergence of $\mathfrak{H}_2$. Convergence of the remaining series can be proved similarly.

Let N_n be the number of nonzero coefficients of the series $\mathfrak{H}_2$ that are between $[2^{\sqrt{n}}] \stackrel{\text{df}}{=} l(n)$ and $[2^{\sqrt{(n+1)}}] \stackrel{\text{df}}{=} l(n+1)$. We have

$$\sum_{j=[2^{\sqrt{n}}]+1}^{[2^{\sqrt{(n+1)}}]}\mu_i^2 \geq \sum_{\substack{j\in A_2^c\\ [2^{\sqrt{n}}]\leq j\leq[2^{\sqrt{(n+1)}}]}}\mu_j^2 \geq c^2 N_n/\ln_2^2 2^{\sqrt{n}}.$$

Since $\sum_{i\geq 0}\mu_i^2 < \infty$ we deduce that $N_n = o(n)$. We also see that $\lambda(n) = l^{-1}(n) \cong (\ln n)^2$. Thus $\mathfrak{H}_2$ converges a.s. if only $\sum_{i\in A_2^c}\mu_i^2(\ln\lambda(i))^2 = \sum_{i\in A_2^c}\mu_i^2(\ln\ln i)^2 < \infty$ by Theorem 2.3.6 of [1].

If we have the opposite case then we apply arguments based on assertion (7) of Lemma 1 and the observation that $\sum_{i\in A_2^c}\mu_i T_i X_{i+1}$ is also an orthogonal series with the same characteristics (i.e. $l(n)$, $\lambda(n)$, and $N(n)$) as $\mathfrak{H}_2$. Further we argue as in the proof of Lemma 3. (Here we have $\sup_{i\in A_2^c}|\mu_i|\ln\ln i < \infty$) with $\ln i$ substituted by $\ln\ln i$ and the Rademacher–Menshov theorem substituted by Theorem 2.3.6 of [1] and show convergence of $\mathfrak{H}_2$.

We treat series $\mathfrak{H}_3, \dots, \mathfrak{H}_k$ in the similar way (here we have $\sup_{i\in A_j^c}|\mu_i|\ln^{(j)} i < \infty$, $j = 3, \dots, k$) and prove their convergence; hence proving convergence of our series. □

III. Applications, extensions, and concluding remarks

1°. *Laws of large numbers.*

(a) Let $\{X_i\}_{i\geq 1}$ be a sequence of SURV. Let us define:

$$Z_n^{(k,\varepsilon)} = \left(\sum_{i=1}^{n} X_i\right) / \sqrt{n\ln(n)\cdots\ln^{(k-1)}(n)(\ln^{(k)}(n))^{1+\varepsilon}}, \qquad k\in\mathbb{N},\ \varepsilon\in\mathbb{R}^{+}.$$

Considering the recursive form of $Z_n^{(k,\varepsilon)}$ and Theorem 4 of [4] we see that $Z_n^{(k,\varepsilon)} \xrightarrow[n\to\infty]{} 0$ a.s. if only series:

$$\sum_{i\geq 2} X_i / \sqrt{i\ln i\cdots\ln^{(k-1)} i(\ln^{(k)} i)^{1+\varepsilon}} \quad \text{converges a.s.}$$

If $\{X_i\}_{i\geq 1}$ did not satisfy conditions (S) and (O), then from the Rademacher–Menshov theorem it would follow that only

$$Z_n^{(1,\varepsilon)} \xrightarrow[n\to\infty]{} 0 \quad \text{a.s. for } \varepsilon > 2,$$

while if $\{X_i\}_{i\geq 1}$ was PSO, then by Theorem 1 we would have

$$\forall k\in\mathbb{N},\ \varepsilon > 2 \qquad Z_n^{(k,\varepsilon)} \xrightarrow[n\to\infty]{} 0 \quad \text{a.s.}$$

(b) Notice that in the proofs of assertions (3), (4), (7) of Lemma 1 and in the proof of Lemma 2, we used orthogonality of the system $\{X_i\}$ only to show that $\sum_{i\geq 1}\nu_i T_i^2 < \infty$ a.s., that is, to prove assertion (2) of Lemma 1. However, if this assertion was assume to be true (that is, $\sum_{i\geq 1}\nu_i T_i^2 < \infty$), then we could use assertions (1), (3), (4), (7) (of Lemma 1).

Hence in particular, if $\{X_i\}_{i\geq 1}$ were standardized (not necessarily uncorrelated) random variables, then according to [4] (Theorem 5) if $\sum_{n\geq 1}(\sqrt{M_n})/n < \infty$ then $(\sum_1^n X_i)/n \xrightarrow[n\to\infty]{} 0$ a.s. $(M_n = E(\sum_1^n X_i)^2/n^2)$. If we assumed, however, that the system $\{X_i\}_{i\geq 1}$ additionally satisfies condition (S), then it would follow from Lemma 2 applied with $\mu_i = 1/(i+1)$, $i\geq 0$ (in fact from the proof of Lemma 2), that condition $\sum_{n\geq 1} M_n/n < \infty$ guarantees that $(\sum_1^n X_i)/n \xrightarrow[n\to\infty]{} 0$ a.s.

2°. *Almost sure convergence.* One might be tempted to think that a PSO orthogonal series converges almost everywhere if only it converges in $\mathbb{L}_2$. We do not know if this conjecture is true. We give two remarks concerning this question. One in favor of this conjecture and one that suggests it is not true.

Remark 8. From Theorem 1 it follows that if $\{X_i\}_{i\geq 1}$ is a PSO system, then

$$\forall k\in\mathbb{N} \qquad E\left(\sup_{i\leq n}|S_i|\right) = o(\ln^{(k)} n).$$

Hence if the sequence $\{S_n\}$ is to be divergent then its converging almost sure subsequence (it always exists) has to be of the form

$$\{S_{2^{(k_n)}(n)}\}_{n\geq 1} \quad \text{where } k_n \uparrow \infty \text{ as } n\to\infty.$$

Remark 9. Notice that every orthogonal series with the orthonormal system $\{X_i\}$ satisfying condition (S) (and also PSO) has the following property. For any a.s. converging subsequence $\{S_{m_j}\}_{j\geq 1}$ of the considered orthogonal series one can construct another converging subsequence $\{S_{n_k}\}_{k\geq 1}$ such that $\{S_{n_k}\} \supset \{S_{m_j}\}$. Further let the nondecreasing sequence of indexes $\{k_i\}$ be defined by $n_{k_i} = m_i$.

Now notice that if the system $\{X_i\}$ satisfied condition (S) then subsequence $\{n_k\}$ would be such that:

$$\sum_{j=k_i+1}^{k_{i+1}} \sqrt{\left(\sum_{q=n_j+1}^{n_{j+1}} \mu_q^2\right)} = O(1) \qquad i \geq 1\,;$$

while if $\{X_i\}$ was a PSO system then $\{n_k\}$ would be defined by:

$$\sum_{j=k_i+1}^{k_{i+1}} \ln^2 j \sum_{q=n_j+1}^{n_{j+1}} \mu_q^2 = O(1) \qquad i \geq 1.$$

To prove these statements we apply Lemma 2 in the first case and Lemma 3 in the second case.

Moreover it follows from these constructions that the numbers of the "condensing" elements, i.e. the number of elements of the sequence $\{S_{n_k}\}$ that lie between the elements of the sequence $\{S_{m_j}\}$ is large. For example in the first case there are more than j elements that lie between S_{m_j} and $S_{m_{j+1}}$, while in the second case we have more than 2^j such elements.

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