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# GEODESICS AND BOUNDED HARMONIC FUNCTIONS ON INFINITE PLANAR GRAPHS

S. NORTHSHIELD

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ABSTRACT. It is shown there that an infinite connected planar graph with a uniform upper bound on vertex degree and rapidly decreasing Green's function (relative to the simple random walk) has infinitely many pairwise finitely-intersecting geodesic rays starting at each vertex. We then demonstrate the existence of nonconstant bounded harmonic functions on the graph.

Let $\mathfrak{g}$ be an infinite, simple, connected, planar graph. $\mathfrak{g}$ also denotes the vertex set of the graph. If two vertices $x$ and $y$ are connected by an edge, we write $xEy$. For a vertex $x$, the degree of $x$ is $d(x) \equiv |\{y \in \mathfrak{g}: yEx\}|$, and we assume:

$$\delta \equiv \sup_{x\in\mathfrak{g}} d(x) < \infty. \tag{1}$$

A finite [infinite] walk $\gamma$ is a sequence $(\gamma(0), \dots, \gamma(n))$ $[(\gamma(0), \gamma(1), \dots)]$ of elements of $\mathfrak{g}$ such that $\gamma(k)E\gamma(k+1)$ for all $0 \le k \le n-1$ [for all $k \ge 0$]. We say that $\gamma$ starts at $\gamma(0)$ and, in the first case, ends at $\gamma(n)$ and has length $n$. Since $\mathfrak{g}$ is connected, we may define a metric:

$$d(x, y) \equiv \inf\{n: n \text{ is the length of a finite walk from } x \text{ to } y\}.$$

A path is a walk whose vertices are distinct. A geodesic $\gamma$ is a path such that $d(\gamma(m), \gamma(n)) = |m-n|$ for all possible $m$ and $n$. For $x \in \mathfrak{g}$, $\Gamma(x, n)$ is the set of geodesics that have length $n$ and start at $x$; $\Gamma(x)$ is the set of geodesics that have infinite length and start at $x$.

The following propositions are useful; the first is easy to prove by a diagonal type argument.

**Proposition 1.** *For all $x \in \mathfrak{g}$, $\Gamma(x) \neq \varnothing$.*

**Proposition 2.** *Given $x$, $y \in \mathfrak{g}$ and $\gamma \in \Gamma(x)$, there exists a $\gamma' \in \Gamma(y)$ such that $\gamma$ and $\gamma'$ eventually coincide.*

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*Proof.* Let $x, y \in \mathfrak{g}$, $\gamma \in \Gamma(x)$. By the triangle inequality, $|d(y, \gamma(n)) - n| = |d(y, \gamma(n)) - d(x, \gamma(n))| \leq d(x, y)$ and, since $d(y, \gamma(n)) - n$ is nonincreasing, $a \equiv \lim_{n\to\infty}[d(y, \gamma(n)) - n] = d(y, \gamma(N)) - N$ for some $N$. Define a path $\gamma'$ where $(\gamma'(0), \dots, \gamma'(d(y, \gamma(N))))$ is a finite geodesic from $y$ to $\gamma(N)$ and, for $k \geq d(y, \gamma(N))$, $\gamma'(k) = \gamma(k-a)$. Then $\gamma' \in \Gamma(y)$. □

Consider the transition probabilities for a Markov chain defined by:

$$p(x, y) \equiv \begin{cases} 1/d(x) & \text{if } yEx, \\ 0 & \text{otherwise.} \end{cases}$$

We denote this chain by $X(0), X(1), \dots$. We let $P^x(\cdot) \equiv P(\cdot|X(0) = x)$ and $E^x(\cdot)$ be the associated expectation operator. Hence, $p(x, y) = P(X(1) = y|X(0) = x) = P^x(X(1) = y)$. $X(\cdot)$ is called the simple random walk on $\mathfrak{g}$.

Let $p^{(n)}(x, y)$ be the $n$-fold convolution of $p$ with itself, and define Green's function as $G(x, y) = \sum_{n\geq 0} p^{(n)}(x, y)$. Probabilistically, $p^{(n)}(x, y) = P^x(X(n) = y)$ and $G(x, y) = E^x(\sum_{n\geq 0}\chi_{\{y\}}(X(n)))$ = the average number of times that the random walk, starting at $x$, hits $y$. It is easy to see that the random walk is transient if and only if $G$ exists (see [2]; his proof for the case when $\mathfrak{g}$ is a tree applies to our case without change). By the strong Markov property,

$$G(x, y) = P^x(\exists n \geq 0 \colon X(n) = y)G(y, y). \tag{2}$$

We assume that Green's function is rapidly decreasing in the sense that

$$\sum_{n\geq 0} n \cdot \sup\{G(x, y) \colon x, y \in \mathfrak{g},\ d(x, y) = n\} < \infty. \tag{3}$$

*Remark.* It is known that the Cheeger condition

$$\exists c > 0 \colon \forall \text{ finite } K \subset \mathfrak{g} \colon \#\{\text{edges from } K \text{ to } K^c\}/|K| \geq c$$

implies $G(x, y) \leq c\varepsilon^{d(x, y)}$ (for some $c$ and $\varepsilon$)—see [1] or [4]. Hence the Cheeger condition implies condition (3).

**Lemma 1.** *For any integer $m \geq 0$, there is an $N(m) \geq 0$ such that if $A$ is the union of $m$ geodesics and $d(x, A) \geq N(m)$, then $P^x(\exists n \colon X(n) \in A) < 1$.*

*Proof.* For any $n \geq 0$, $x \in \mathfrak{g}$, let $S(x, n)$ and $B(x, n)$ be the metric sphere and ball respectively with centers $x$ and radii $n$. If $\gamma$ is a geodesic, then $|\gamma \cap S(x, n)| \leq |\gamma \cap B(x, n)| \leq 2n + 1$. Hence $|A \cap S(x, n)| \leq (2n+1)m$ and we get

$$\begin{aligned} P^x(\exists n \geq 0\colon X(n) \in A) &\leq \sum_{y \in A} P^x(\exists n \geq 0\colon X(n) = y) \\ &= \sum_{y \in A} \frac{G(x, y)}{G(y, y)} \quad \text{(by (2))} \\ &\leq \sum_{y \in A} G(x, y) \quad (\text{since } G(y, y) \geq 1) \\ &\leq \sum_{n \geq d(x, A)} |A \cap S(x, n)| \cdot \sup\{G(x, y)\colon d(x, y) = n\} \\ &\leq m \sum_{n \geq d(x, A)} (2n+1) \cdot \sup\{G(x, y)\colon d(x, y) = n\}. \end{aligned}$$

By (3), choose $N(m)$ so that $m \sum_{n \geq N(m)} (2n+1) \cdot \sup\{G(x, y)\colon d(x, y) = n\} < 1$. □

**Lemma 2.** *For any* $K \subset \mathfrak{g}$, *if* $\inf_{x \in \mathfrak{g}} P^x(\limsup_{n \to \infty}(X(n) \in K)) < 1$, *then* $\sup_{x \in \mathfrak{g}} d(x, K) = \infty$.

*Proof*. By condition (1), for any $y \in K$ and $x \in \mathfrak{g}$,

$$P^x(\exists n\colon X(n) \in K) \geq P^x(X(d(x, y)) = y) \geq (1/\delta)^{d(x, y)}.$$

Thus, if $\sup_{x \in \mathfrak{g}} d(x, K) < \infty$, then $\inf_{x \in \mathfrak{g}} P^x(\exists n\colon X(n) \in K) > 0$ and, therefore, $\inf_{x \in \mathfrak{g}} P^x(\limsup_{n \to \infty}(X(n) \in K)) = 1$. □

**Theorem 1.** *For any* $x \in \mathfrak{g}$, *there are infinitely many geodesic rays* $\gamma_1, \gamma_2, \ldots$ *starting at* $x$ *such that if* $i \neq j$, *then* $|\gamma_i \cap \gamma_j| < \infty$.

*Proof*. We construct such a family inductively. There is always one geodesic ray starting at $x$ (Proposition 1). Suppose $\gamma_1, \gamma_2, \ldots, \gamma_m \in \Gamma(x)$ such that if $i \neq j$, then $|\gamma_i \cap \gamma_j| < \infty$. Let $\partial A = \bigcup_{i=1}^n \gamma_i$. By Proposition 2, it is enough to show that there exists a geodesic ray $\gamma$ such that $\gamma \cap \partial A = \varnothing$. Therefore, by the diagonal method of Proposition 1, it is enough to show that there exists $z \in \mathfrak{g}$ such that for all $k$, there exists $\gamma_k \in \Gamma(z, k)$ so that $\gamma_k \cap \partial A = \varnothing$.

Let $A = \mathfrak{g} \setminus \partial A$ and $N = N(m+2)$ where $N(\cdot)$ is as in Lemma 1. As in the proof of Lemma 1, $\sum_{y \in \partial A} G(x, y) < \infty$ and so

$$P^x\left(\limsup_{k \to \infty}(X(k) \in \partial A)\right) = 0.$$

By Lemma 2, we can choose $z \in A$ such that $d(z, \partial A) \geq N$.

Suppose that there exists $n$ such that for all $\gamma \in \Gamma(z, n)$, $\gamma \cap \partial A \neq \varnothing$. We show that this leads to a contradiction—we show that this implies the existence of two geodesic segments $\gamma_t^*$ and $\gamma_u^*$ such that:

(a) $d(z, \gamma_t^* \cup \gamma_u^*) \geq N$ and

(b) every infinite path starting at $z$ hits $\gamma_t^* \cup \gamma_u^* \cup \partial A$.

By Lemma 1, condition (a) implies $P^z(\exists j\colon X(j) \in \gamma_t^* \cup \gamma_u^* \cup \partial A) < 1$ whereas condition (b) implies $P^z(\exists j\colon X(j) \in \gamma_t^* \cup \gamma_u^* \cup \partial A) = 1$.

For each $y \in S(z, n)$, choose $\gamma_y \in \Gamma(z, n)$. In addition, we choose these geodesics so that $\bigcup \gamma_y$ is a tree. For any $y \in S(z, n)$, let $\gamma_y^* = (\gamma_y(\eta), \dots, \gamma_y(n))$ where $\eta = \max\{j \le n : \gamma_y(j) \in \partial A\}$. Note that for any $t$, $u \in S(z, n)$, condition (a) holds. Let $Z = \{y \in S(z, n)$: there exists an infinite path in $A$, starting at $z$, which last hits $B(z, n)$ at $y\}$. $Z$ is nonempty by choice of $N$ and $z$. For $Y \subset Z$, let $C(Y)$ be the connected component of $B(z, n) \setminus (\partial A \cup \bigcup_{y \in Y} \gamma_y^*)$ which contains $z$.

We claim that $C(Z) = C(\{t, u\})$ for some $t$, $u \in Z$. If so, then condition (b) holds for $t$ and $u$. To prove this claim, it is enough to show that if $t$, $u$, $v$ are distinct elements of $Z$, then $C(\{t, u, v\}) = C(\{t^\cdot, u^\cdot\})$ for some $t^\cdot$, $u^\cdot \in \{t, u, v\}$.

Let $t$, $u$, $v$ be distinct elements of $Z$, and let $\rho$, $\sigma$, $\tau$ be infinite paths in $A$ starting at $z$ which last hit $B(z, n)$ at $t$, $u$, $v$ respectively. Since $\partial A$ is connected, $\partial A$ is in one of the components of $G \setminus (\rho \cup \sigma \cup \tau)$. By planarity, without loss of generality, any path from $t$ to $\partial A$ must hit $\sigma \cup \tau$. Define $\rho^*(j) = \rho(j + M + 1)$ where $M = \max\{k : \rho(k) \in B(z, n)\}$. Then the complement of $\partial A \cup \gamma_t^* \cup \rho^*$ contains two components, say $B$ and $C$, such that $u \in B$, $v \in C$, and, without loss of generality, $z \in B$. Then, any path contained in $A$ from $z$ to $v$ must hit either $\gamma_t^*$ or $\rho^*$. Since $\rho^* \cap B(z, n) = \varnothing$, $C(\{t, u, v\}) = C(\{t, u\})$. $\square$

A function $f\colon \mathfrak{g} \to \mathbb{R}$ is harmonic if and only if $\sum_{y : yEx} f(y) = d(x) f(x)$ for all $x$. In particular, since $\liminf_{k\to\infty}(X(k) \in A)$ is invariant under the Markov shift, $f(x) \equiv P^x(\liminf_{k\to\infty}(X(k) \in A)) = pf(x)$ and so $f$ is bounded and harmonic. We use an idea similar to one Kendall uses in the case of Brownian motion on manifolds [3] to find a set $A$ so that $P^*(\liminf_{k\to\infty}(X(k) \in A))$ is nonconstant.

**Theorem 2.** *There are nonconstant, bounded, harmonic functions on* $\mathfrak{g}$.

*Proof*. Let $N = N(2)$ where $N(\cdot)$ is as in Lemma 1. Fix $x \in \mathfrak{g}$ and, by Theorem 1, choose $4N$ rays $\gamma_1, \gamma_2, \dots, \gamma_{4N} \in \Gamma(x)$ whose pairwise intersections are finite. Without loss of generality, these geodesics are numbered in a clockwise fashion (we may do this since $\mathfrak{g}$ is planar). Let $M$ be such that $i \ne j$ implies $(\gamma_i \cap \gamma_j) \setminus B(x, M) = \varnothing$. Let $C = \gamma_1 \cup \gamma_{2N}$, $u = \gamma_N(M+N)$, $v = \gamma_{3N}(M+N)$, and $A$ and $B$ be the connected components of $\mathfrak{g} \setminus C$ containing $u$ and $v$ respectively. By Lemma 1, since $d(u, C) \ge N$ and $d(v, C) \ge N$,

$$P^u\left(\liminf_{k\to\infty}(X(k) \in A)\right) \ge P^u(\forall j : X(j) \notin C) > 0$$

and

$$P^v\left(\limsup_{k\to\infty}(X(k) \in A)\right) \le P^v(\exists j : X(j) \in C) < 1\,.$$

By Lemma 2,

$$\sup_{w \in \mathfrak{g}} d(w, A) = \infty\,.$$

Since, for $w \in B$,

$$P^w\left(\liminf_{k\to\infty}(X(k) \in A)\right) \le P^w(\exists j\colon X(j) \in C)$$
$$\le 2c \sum_{n \ge d(w\,,A)} (2n+1) \cdot \sup\{G(x\,,y)\colon d(x\,,y) = n\}$$

(as in the proof of Lemma 1), and since $d(w\,,A)$ is unbounded,

$$\inf_w P^w\left(\liminf_{n\to\infty}(X(n) \in A)\right) = 0$$

and so is not constant. □

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Department of Mathematics, State University of New York College at Plattsburgh, Plattsburgh, New York 12901