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LINKING AND THE SHADOWING PROPERTY FOR PIECEWISE MONOTONE MAPS

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ABSTRACT. A necessary and sufficient condition for a continuous uniformly piecewise linear map of a compact interval to have the Shadowing Property is that its turning point set be linked.

1. INTRODUCTION

Given δ a positive number, a sequence $\{x_j\}$ of points in a metric space (X, d) is called a *δ-pseudo-orbit* of a map $f: X \to X$ if

$$d(f(x_j), x_{j+1}) \leq \delta \quad \text{for every } j.$$

A point $x \in X$ *ε-shadows* $\{x_j\}$ if

$$d(f^j(x), x_j) < \varepsilon \quad \text{for every } j.$$

We say f has the *Shadowing Property* on X if: *For every $\varepsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo-orbit $\{x_j\}$ of f in X can be ε-shadowed by a point in X.*

Earlier work on pseudo-orbits and shadowing includes Anosov [A], Franke–Selgrade [FS], Bowen [Bo], Walters [W], and Coven–Kan–Yorke [CKY].

In this paper, we study maps that are conjugate to continuous uniformly piecewise linear maps of a compact interval to itself, and we show for this kind of maps, that the Shadowing Property is completely determined by the dynamics of their turning point sets. Our main result is:

Theorem. *Suppose f is a map that is conjugate to a continuous uniformly piecewise linear map of a compact interval to itself. Then f has the Shadowing Property if and only if it has the Linking Property.*

In §2, we define *ε-linking* and *linking*, and discuss a linking property that is preserved by conjugacy and iteration for uniformly piecewise linear maps of

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a compact interval to itself. In §3, we prove the theorem, and in §4 we give several examples for this result.

2. Preliminaries

$\langle a, b\rangle$ is used to denote a closed interval with endpoints a and b, regardless of the order of a and b. If $f: I \to I$ is continuous, defined on a compact interval I, and takes its local extrema at finitely many points, then f is called *piecewise monotone*. Those points at which f takes local extremum are called *turning points* of f, and the set of all turning points of f is denoted by $C(f)$. Notice that our definition includes the endpoints of I as turning points.

The *orbit* of x under f is defined by $\mathrm{orb}_f(x) = \{x, f(x), \dots, f^n(x), \dots\}$. The *$\omega$-limit set* of x is defined by $\omega(x, f) = \{y | f^{n_k}(x) \to y$ for a sequence $n_k \nearrow +\infty\}$. Let $B(x, r)$ denote the closed ball of radius r about x in a metric space, then in a compact interval I we have $B(x, r) = [x-r, x+r] \cap I$.

The following lemma, proven by Coven, Kan, and Yorke in [CKY], is used in the proof of our main result.

Lemma 0 ([CKY, Lemma 2.4]). *Suppose f is a continuous map of a compact metric space (X, d). f has the Shadowing Property provided there is a constant $\lambda \geq 1$ such that for every $\varepsilon > 0$, there are $\delta > 0$, a positive integer N, and for each $x \in X$ a positive integer $n = n(x) \leq N$ so that*

$$(1)\quad f[B(f^n(x), \varepsilon + \delta)] \subset \{f^{n+1}(y) : d(x, y) \leq \varepsilon \text{ and } d(f^j(x), f^j(y)) \leq \lambda\varepsilon \text{ for } 1 \leq j \leq n\}.$$

A. Linkings and the Linking Property

Definition 1. Suppose f is a map of a compact metric space (X, d) to itself and ε is a positive number. We say *a point $x \in X$ is ε-linked to a point $y \in X$ by f* (or there exists an *ε-linking from x to y for f*) if there exists an integer $m \geq 1$ and a point $z \in B(x, \varepsilon)$ such that $f^m(z) = y$ and

$$d(f^j(x), f^j(z)) \leq \varepsilon \quad \text{for } 0 \leq j \leq m.$$

We say $x \in X$ *is linked to* $y \in X$ *by* f if x is ε-linked to y by f for every $\varepsilon > 0$. We say a subset C of X *is linked* (resp. *ε-linked*) *by* f if every point $c \in C$ is linked (ε-linked) to some point in C by f.

The set of points to which x can be linked (resp. ε-linked) by f is denoted by $\mathscr{A}(x, f)$ (resp. $\mathscr{A}_\varepsilon(x, f)$).

Remark 1. x is linked to y by f if and only if for any sequence $\{\varepsilon_k\}$ with $\varepsilon_k \to 0$, there are a sequence of integers m_k and points $z_k \in B(x, \varepsilon_k)$ so that $f^{m_k}(z_k) = y$ and

$$d(f^j(x), f^j(z_k)) \leq \varepsilon_k \quad \text{for } 0 \leq j \leq m_k.$$

In particular, when $y \notin \mathrm{orb}_f(x)$, we have $m_k \to \infty$ and $f^{m_k}(x) \to y$. This implies that $y \in \omega(x, f)$. Moreover,

$$\mathrm{orb}_f(x) \subset \mathscr{A}(x, f) \subset \omega(x, f) \cup \mathit{orb}_f(x) \quad \text{for every } x \in X.$$

Remark 2. It is easy to see if X and Y are compact spaces and $f: X \to X$ is topologically conjugate to $g: Y \to Y$ by $h: X \to Y$, then $h[\mathscr{A}(x, f)] = \mathscr{A}(h(x), g)$.

Remark 3. Given $x \in X$, for every $y \in \omega(x, f)$ we have $\mathscr{A}(y, f) \subset \omega(x, f)$.

Linking Property. *For a piecewise monotone map f of a compact interval I to itself, let $C(f)$ be the turning point set of f. We say f has the Linking Property if $C(f)$ is linked by f.*

The following lemmas are obvious (cf. [Ch]).

Lemma 1. *The Linking Property is preserved by a topological conjugacy.*

Lemma 2. *Suppose f is a map of a compact metric space (X, d) and C is a finite subset of X. Then there is $\varepsilon > 0$ such that for every $c \in C$, if c is ε-linked to some point $b \in C$, then c is linked to b.*

In particular, there is $\varepsilon > 0$ such that if C is ε-linked by f, then C is linked by f.

B. Uniformly piecewise linear maps

Definition 2. Suppose f is a continuous piecewise monotone map on I. If $x \notin C(f)$, we define

$$\tau(x) = \begin{cases} +1 & \text{if } f \text{ is increasing at } x, \\ -1 & \text{if } f \text{ is decreasing at } x, \end{cases}$$

and define a *signature system* for each turning point of f as follows:

Suppose $c \in C(f)$, and define $\sigma_1(c)$ by

$$\sigma_1(c) = \begin{cases} -1 & \text{if } f \text{ takes a local maximum at } c, \\ +1 & \text{if } f \text{ takes a local minimum at } c. \end{cases}$$

Inductively definite $\sigma_n(c)$ by

$$\sigma_{n+1}(c) = \begin{cases} \tau(f^n(c))\sigma_n(c) & \text{if } f^n(c) \notin C(f), \\ \sigma_1(f^n(c)) & \text{if } f^n(c) \in C(f). \end{cases}$$

A piecewise monotone map f on $[a, b]$ is *uniformly piecewise linear* if there are $s > 1$ and $\{\alpha_i\}_{i=1}^m$ such that

$$f(x) = \alpha_i \pm sx \quad \text{for } x \in [a_{i-1}, a_i],$$

where $a_0 = a < a_1 < \cdots < a_m = b$ are the turning points of f and the sign involved depends only on $i = 0, 1, \ldots, m$. The constant s is called the *slope of* f, denoted by $\mathrm{slope}(f) = s$. In [P] it was shown that $h(f) = \log s$ where h denotes the topological entropy.

Remark 4. The eventual turning points of a uniformly piecewise linear map f are dense: if $x \neq y$, then there exists j such that $\mathrm{Int}\langle f^j(x), f^j(y)\rangle \cap C(f) \neq \varnothing$.

Remark 5. Suppose f is uniformly piecewise linear and $c \in C(f)$. The following assertions follow from the definitions:

(1) If $\varepsilon > 0$ is small enough that $B(c, 2\varepsilon) \cap C(f) = \{c\}$, then for $x \in B(c, \varepsilon)$

$$f[B(x, \varepsilon)] = \langle f(c), f(x) + \sigma_1(c)s\varepsilon \rangle;$$

(2) If $C(f) \cap \operatorname{Int}\langle f^j(c), f^j(x)\rangle = \varnothing$ for $j = 0, \dots, l-1$, then

$$f^l(x) = f^l(c) + \sigma_l(c)s^l|x - c|.$$

Definition 3. Define

$$B_k(x, \varepsilon) = \bigcap_{j=0}^{k} f^{-j}[B(f^j(x), \varepsilon)]$$
$$= \{y \in I \mid |f^j(y) - f^j(x)| \le \varepsilon \text{ for } j = 0, \dots, k\}$$

and

$$D_k(x, \varepsilon) = f^k[B_k(x, \varepsilon)].$$

Remark 6. It is easy to see from the definitions that x is ε-linked to y iff $y \in D_k(x, \varepsilon)$ for some $k \ge 1$, i.e.

$$\mathscr{A}_\varepsilon(x, f) = \bigcup_{k=1}^{\infty} D_k(x, \varepsilon).$$

The following lemmas are easily proved.

Lemma 3. *Suppose f is a continuous uniformly piecewise linear map, $c \in C(f)$ and $\varepsilon > 0$ is small enough that $C(f) \cap B(c, 2\varepsilon) = \{c\}$. If $\operatorname{Int} D_j(c, \varepsilon) \cap C(f) = \varnothing$ for $j = 1, \dots, k-1$, then*

(i) $D_k(c, \varepsilon) = \langle f^k(c), f^k(c) + \sigma_k(c)\varepsilon \rangle$;

(ii) *for* $x \in B(c, \varepsilon/s^k)$,

$$f^k(x) = f^k(c) + \sigma_k(c)s^k|x - c|;$$

(iii) *if* $x \in B(c, \varepsilon/s^k)$ *also satisfies* $C(f) \cap \operatorname{Int} D_j(x, \varepsilon) = \varnothing$ *for* $j = 1, \dots, k-1$, *then*

$$D_k(x, \varepsilon) = \langle f^k(c), f^k(x) + \sigma_k(c)\varepsilon \rangle.$$

Lemma 4. *Suppose f is a continuous uniformly piecewise linear map on I and n is any given positive integer. Then f has the Linking Property if and only if f^n does.*

Lemma 5. *If f is a continuous uniformly piecewise linear map, then the following statements are equivalent:*

(i) *f has the Linking Property;*

(ii) *For every $\varepsilon > 0$ and $c \in C(f)$, there are an integer $n \ge 1$ and a point $b \in C(f)$ such that $|f^n(c) - b| \le \varepsilon$, and either $f^n(c) = b$ or $|f^n(x) - b|$ has a local maximum at $x = c$.*

3. Main result and proof

In this section, we prove the following theorem:

Theorem. *Suppose f is a map that is conjugate to a continuous uniformly piecewise linear map of a compact interval to itself. Then f has the Shadowing Property if and only if it has the Linking Property.*

We need the following facts:

Fact 1. The Shadowing Property on a compact space is preserved by a topological conjugacy.

Fact 2. Suppose f is a continuous map of a compact metric space to itself. Then for any integer $n \geq 1$, f has the Shadowing Property if and only if f^n does.

The direction "$\Leftarrow$" of Fact 2 can be proven by the uniform continuity of f and the inequality of the metric d for any $0 \leq l \leq n-1$ and $k \geq 0$

$$d(f^{kn+l}(y), x_{kn+l}) \leq d(f^{kn+l}(y), f^l(x_{kn})) + \sum_{j=0}^{l-1} d(f^{l-j}(x_{kn+j}), f^{l-j-1}(x_{kn+j+1})).$$

According to Fact 1 and Lemma 1, we can assume that f is uniformly piecewise linear without loss of generality. According to Fact 2 and Lemma 4, the Shadowing Property and the Linking Property for f are equivalent respectively to these properties for any iterate of f. Since $s > 1$, we have $\mathrm{slope}(f^n) = s^n > 2$ for $n > \ln 2/\ln s$. Replacing f by f^n if necessary, we can assume $s = \mathrm{slope}(f) > 2$ in the following proof.

1. **The proof that Shadowing Property $\Rightarrow$ Linking Property**

Lemma 6. *If f is a continuous uniformly piecewise linear map of I to itself with the Shadowing Property, then for every $c \in C(f)$*

(2) $$\omega(c, f) \cap C(f) \neq \varnothing.$$

Proof. Suppose $\omega(c, f) \cap C(f) = \varnothing$ for some $c \in C(f)$. Then there exist $\varepsilon_0 > 0$ and $l > 0$ such that

$$f^k(c) \notin U_{\varepsilon_0}(C(f)) \quad \text{for all } k \geq l,$$

where $U_{\varepsilon_0}(C(f)) = \bigcup\{B(c, \varepsilon_0) | c \in C(f)\}$. Choose $\varepsilon_1 \in (0, \varepsilon_0)$ small enough that

$$\mathrm{Int}\langle f^j(c), f^j(c) + \sigma_j(c)s^j\varepsilon_1\rangle \cap C(f) = \varnothing \quad \text{for all } 0 \leq j \leq l.$$

For any $\delta \in (0, \varepsilon_1/s^l)$, define a δ-pseudo-orbit as follows:

$$x_j = \begin{cases} f^j(c) & \text{for } 0 \leq j < l, \\ f^{j-l}[f^l(c) - \sigma_l(c)\delta] & \text{for } j \geq l. \end{cases}$$

For a small $\delta > 0$, if $\{x_j\}$ is ε_1-shadowed by a point y, then by Remark 5

$$f^l(c) = f^l(y) - \sigma_l(c)s^l|y - c| \in \langle x_l, f^l(y)\rangle.$$

This implies that $\langle x_l, f^l(y)\rangle \cap C(f) = \varnothing$, since $f^l(c) \notin U_{\varepsilon_0}(C(f))$ and $|x_l - f^l(y)| \leq \varepsilon_0$. It follows that $f^{l+1}(c) \in f^l[\langle x_l, f^l(y)\rangle] = \langle x_{l+1}, f^{l+1}(y)\rangle$. If we assume that $f^j(c) \in \langle x_j, f^j(y)\rangle$ for some $j \geq l$, then since $f^j(c) \notin U_{\varepsilon_0}(C(f))$ and $|x_j - f^j(y)| \leq \varepsilon_0$, we have $\langle x_j, f^j(y)\rangle \cap C(f) = \varnothing$. Furthermore, $f^{j+1}(c) \in f[\langle x_j, f^j(y)\rangle] = \langle x_{j+1}, f^{j+1}(y)\rangle$. So, by induction, we have $f^j(c) \in \langle x_j, f^j(y)\rangle$ and $\langle x_j, f^j(y)\rangle \cap C(f) = \varnothing$ for every $j \geq l$. By Remark 4, we must have $f^l(y) = x_l = f^l(c) - \sigma_l(c)\delta$. But by Remark 5, we have $f^l(y) \in f^l[B(c, \delta)] = \langle f^l(c), f^l(c) + \sigma_l(c)s^l\delta\rangle$, and this is a contradiction. □

Hypothesis. Until the end of the proof of Proposition 11, we assume that f is uniformly piecewise linear with $\operatorname{slope}(f) = s$ and has the Shadowing Property.

By Lemma 6, (2) holds for every $c \in C(f)$. Now given a point $c \in C(f)$, we need to show that c is linked to a point of $C(f)$. By (2), there exists $b \in \omega(c, f) \cap C(f)$.

Lemma 7. *Suppose $c \in C(f)$ and $b \in \omega(c, f) \cap C(f)$. If c is not linked to b, then b is linked to some point $C(f)$.*

Proof. If b is not linked to any point of $C(f)$, then there is $\tau > 0$ such that

$$D_j(b, \tau) \cap C(f) = \varnothing \quad \text{for all } j \geq 1. \tag{3}$$

Since f has the Shadowing Property, for any $\varepsilon \in (0, \tau)$ there is $\delta > 0$ such that every δ-pseudo-orbit of f can be ε/s-shadowed. Because $b \in \omega(c, f)$, one can find an integer $l > 0$ such that $|f^l(c) - b| \leq \delta$. Let

$$x_j = \begin{cases} f^j(c) & \text{for } 0 \leq j < l, \\ f^{j-l}(b) & \text{for } j \geq l. \end{cases}$$

Then $\{x_j\}$ is a δ-pseudo-orbit of f. If y is a point that ε/s-shadows $\{x_j\}$, then

$$|f^j(y) - f^j(c)| \leq \varepsilon/s \quad \text{for } 0 \leq j < l \tag{4}$$

and

$$|f^j(y) - f^{j-l}(b)| \leq \varepsilon/s < \tau \quad \text{for all } j \geq l. \tag{5}$$

Notice that (3) and (5) imply that $f^l(y) = b$ and (4) implies that

$$|f^j(y) - f^j(c)| \leq \varepsilon \quad \text{for } j = 0, \dots, l.$$

So c is ε-linked to b. For ε small enough, by Lemma 2, we have $b \in \mathscr{A}(c, f)$, contradicting the assumption. □

Lemma 8. *If $c \in C(f)$ and $[\mathscr{A}(b, f) - \operatorname{orb}_f(b)] \cap C(f) \neq \varnothing$ for some $b \in \omega(c, f) \cap C(f)$, then c is linked to a point of $C(f)$.*

Proof. Let $\varepsilon_0 > 0$ be as in Lemma 2, that is, if c is ε_0-linked to $\tilde{c} \in C(f)$ then c is linked to $\tilde{c}$. Since $C(f)$ is finite, so is $C(f) \cup f[C(f)]$. Let $\eta > 0$

be the smallest distance between any two points in $C(f) \cup f[C(f)]$. For any $\varepsilon \in (0, \min\{\varepsilon_0, \eta\})$, it suffices to show

$$D_j(c, \varepsilon) \cap C(f) \neq \varnothing \quad \text{for some } j \geq 1.$$

Consider $l = \min\{k \mid \operatorname{Int} D_k(b, \varepsilon) \cap C(f) \neq \varnothing\}$. By assumption, l is finite, and by the choice of η, $l \geq 2$. Let $b' \in \operatorname{Int} D_l(b, \varepsilon) \cap C(f)$, then the number $\varepsilon^* = \min\{\varepsilon/2, |f^l(b) - b'|\}$ is strictly positive. It follows from the Shadowing Property of f that there is $\delta > 0$ such that every δ-pseudo-orbit of f can be (ε^*/s)-shadowed. By the assumption that $b \in \omega(c, f)$, there is an integer $L > 0$ such that $|f^L(c) - b| < \delta$. Define

$$x_j = \begin{cases} f^j(c) & \text{for } 0 \leq j < L, \\ f^{j-L}(b) & \text{for } j \geq L. \end{cases}$$

$\{x_j\}$ is a δ-pseudo-orbit. Suppose y (ε^*/s)-shadows $\{x_j\}$; then

$$(6) \qquad |f^j(c) - f^j(y)| \leq \varepsilon^*/s \quad \text{for } 0 \leq j < L$$

and

$$(7) \qquad |f^{j-L}(b) - f^j(y)| \leq \varepsilon^*/s \quad \text{for all } j \geq L.$$

If $D_j(c, \varepsilon) \cap C(f) \neq \varnothing$ for some j with $1 \leq j < L+l$, we are done. Otherwise, by Lemma 3 we have

$$(8) \qquad D_j(c, \varepsilon) = \langle f^j(c), f^j(c) + \sigma_j(c)\varepsilon\rangle \quad \text{for } j = 1, \dots, L+l.$$

By (6) and (7), we have

$$|f^L(c) - f^L(y)| < \varepsilon^* \leq \varepsilon/2 \quad \text{and} \quad |f^L(y) - b| < \varepsilon^* \leq \varepsilon/2.$$

So

$$(9) \qquad |f^L(c) - b| \leq |f^L(c) - f^L(y)| + |f^L(y) - b| < \varepsilon.$$

By assumption, we have $b \notin D_L(c, \varepsilon)$; it follows from (8) and (9) that

$$\sigma_L(c) = \operatorname{sign}[f^L(c) - b].$$

By the choice of y, we have $f^{L-1}(y) \in D_{L-1}(c, \varepsilon/s)$. Furthermore, $f^L(y) \in D_L(c, \varepsilon) = \langle f^L(c), f^L(c) + \sigma_L(c)\varepsilon\rangle$. This implies that $f^L(c) \in \langle f^L(y), b\rangle$.

By the choice of l and y, for $1 \leq j < l$ we have

$$\begin{aligned} f^{L+j}(c) \in \langle f^{L+j}(y), f^j(b)\rangle &\subset D_j(b, \varepsilon^*/s) \\ &= \langle f^j(b), f^j(b) + \sigma_j(b)\varepsilon^*/s\rangle. \end{aligned}$$

It follows from $\operatorname{Int} D_j(b, \varepsilon) \cap C(f) = \varnothing$ that $\sigma_{L+j}(c) = \sigma_j(b)$ for $j = 1, \dots, l-1$ and $|f^{L+l-1}(c) - f^{l-1}(b)| \leq |f^{L+l-1}(y) - f^{l-1}(b)| \leq \varepsilon^*/s$. Since $\langle f^{L+l-1}(c), f^{l-1}(b)\rangle \cap C(f) \subset D_{l-1}(b, \varepsilon) \cap C(f) = \varnothing$, by Lemma 3 and the choice of ε^*, we have

$$(10) \qquad |f^l(b) - f^{L+l}(c)| \leq \varepsilon^* \leq |f^l(b) - b'|$$

and

$$\sigma_{L+l}(c) = \sigma_l(b). \tag{11}$$

Since $b' \in D_l(b, \varepsilon) = \langle f^l(b), f^l(b) + \sigma_l(b)\varepsilon \rangle$, (10) and (11) imply that $b' \in D_{L+l}(c, \varepsilon)$. □

Lemma 9. *If $c \in C(f)$ and $\omega(c, f) \cap C(f)$ contains a periodic point, then c is linked to this periodic point.*

Proof. Suppose $b \in \omega(c, f) \cap C(f)$ is a periodic point of period m. Since $C(f)$ is finite, we can choose $\sigma > 0$ such that

$$\left[\bigcup_{i=0}^{m} B(f^i(b), \sigma)\right] \cap C(f) \subset \operatorname{orb}_f(b).$$

Thus, for any $\varepsilon \in (0, \sigma]$

$$\operatorname{Int} D_j(b, \varepsilon) \cap C(f) = \emptyset \quad \text{for all } j \geq 1. \tag{12}$$

Choose $\varepsilon \in (0, \sigma]$ as in Lemma 2, then by the Shadowing Property of f there is $\delta > 0$ such that every δ-pseudo-orbit of f can be (ε/s)-shadowed. By the assumption $b \in \omega(c, f)$, there is an integer $L > 0$ such that $|f^L(c) - b| < \delta$. Define

$$x_j = \begin{cases} f^j(c) & \text{for } 0 \leq j < L, \\ f^{j-L}(b) & \text{for } j \geq L. \end{cases}$$

$\{x_j\}$ is a δ-speudo-orbit of f. Suppose y (ε/s)-shadows $\{x_j\}$; then by (12) we have $f^L(y) = b$. Notice that $f^{L-1}(y) \in D_{L-1}(c, \varepsilon/s)$ implies that $b = f^L(y) \in D_L(c, \varepsilon)$. We are done by Lemma 2. □

Lemma 10. *If $c \in C(f)$ and $\omega(c, f) \cap C(f)$ is linked by f, then c is linked to some point of $C(f)$.*

Proof. Since $\omega(c, f) \cap C(f)$ is linked by f, for every $b \in \omega(c, f) \cap C(f)$ we have

$$\mathscr{A}(b, f) \cap \omega(c, f) \cap C(f) \neq \emptyset. \tag{13}$$

If $[\mathscr{A}(b, f) - \operatorname{orb}_f(b)] \cap C(f) \neq \emptyset$ for some $b \in \omega(c, f) \cap C(f)$, then by Lemma 8, we are done, otherwise, for every $b \in \omega(c, f) \cap C(f)$

$$\begin{aligned} &\mathscr{A}(b, f) \cap \omega(c, f) \cap C(f) \\ &\quad = \{b' \in \omega(c, f) \cap C(f) | b' = f^l(b) \text{ for some } l \geq 1\}. \end{aligned} \tag{14}$$

Let b_0 be any point of $\omega(c, f) \cap C(f)$, then by (13) and (14), there is $l_1 \geq 1$ such that $b_1 = f^{l_1}(b_0) \in \omega(c, f) \cap C(f)$. Replacing b_0 by b_1, we can find $l_2 \geq 1$ such that $f^{l_2}(b_1) = f^{l_1+l_2}(b_0) \in \omega(c, f) \cap C(f)$. By induction, for every $k \geq 1$, we have $\{l_k\}$, $l_k \geq 1$ and

$$b_k = f^{l_1+\cdots+l_k}(b_0) \in \omega(c, f) \cap C(f).$$

Since $C(f)$ is finite, there are k', k'' with $k' < k''$ satisfying $b_{k'} = b_{k''}$. It follows that $b_{k'} \in \omega(c, f) \cap C(f)$ is a periodic point of f and by Lemma 9, we are done. □

Proposition 11. *If f is continuous uniformly piecewise linear, then the Shadowing Property for f implies the Linking Property for f.*

Proof. Suppose f has the Shadowing Property and let c be any point of $C(f)$; we want to show that c is linked to some point of $C(f)$. By Lemma 6, we have $\omega(c, f) \cap C(f) \neq \varnothing$.

If c is not linked to any point of $C(f)$, then by Lemma 7, every point $b \in \omega(c, f) \cap C(f)$ must be linked to a point $b' \in C(f)$, and by Remark 3, we have $b' \in \omega(c, f)$. Hence, $\omega(c, f) \cap C(f)$ is linked by f. By Lemma 10, we have a contradiction. □

2. **The proof that Linking Property ⇒ Shadowing Property.**

The basic idea of our proof is the same as that given by Coven–Kan–Yorke [CKY] for tent maps. Here we just give a sketch of the proof to show the reader how the idea works for uniformly piecewise linear maps with more than one internal turning points.

Pick $c \in C(f)$, $x \in I$, and $\varepsilon > 0$, which remain fixed until the end of the section. It is not hard to prove the following (cf. [Ch]):

Lemma 12. *Suppose for an integer $L \geq 3$,*

(1) *$D_j(c, \varepsilon) \cap C(f) = \varnothing$ for $1 \leq j < L$;*
(2) *$|x - c| \leq \varepsilon / s^{L-1}$; and*
(3) *the smallest k for which $D_k(x, \varepsilon) \cap C(f) \neq \varnothing$ satisfies $3 \leq k \leq L$ and $D_{k-1}(x, s\varepsilon) \cap C(f) = \varnothing$.*

Then

$$f[B(f^k(x), s\varepsilon)] \subset f^3[D_{k-2}(x, \varepsilon)].$$

Lemma 13. *If $M \geq 1$ is the smallest integer with $D_m(c, \varepsilon) \cap C(f) \neq \varnothing$, then for any point $x \in B(c, \varepsilon / s^M)$ there is an integer k, with $1 \leq k \leq M$, such that*

$$D_k(x, \varepsilon) \cap C(f) \neq \varnothing.$$

Lemma 14. *Suppose for an integer $L \geq 1$,*

(1) *$D_j(c, \varepsilon) \cap C(f) = \varnothing$ and $D_j(x, \varepsilon) \cap C(f) = \varnothing$ for $j = 1, \ldots, L-1$;*
(2) *$\varepsilon / s^L < |x - c| \leq \varepsilon / s^{L-1}$.*

Then $D_L(x, \varepsilon) = B(f^L(x), \varepsilon)$.

Based on Lemmas 12–14, we have

Proposition 15. *If f is a continuous uniformly piecewise linear map of I to itself, then the Linking Property for f implies the Shadowing Property for f.*

Proof. Since $C(f) \cup f[C(f)] \cup f^2[C(f)]$ is finite, let $\eta > 0$ be the smallest distance between any two points in $C(f) \cup f[C(f)] \cup f^2[C(f)]$. For any $\varepsilon \in (0, \eta/(2s^4))$, let $\lambda = s^2$ and $\delta = (s-1)\varepsilon$, then by Lemma 0 we need only to show that there is $N > 0$ and for each $x \in I$ an integer $n = n(x) \leq N$ so that

$$f[B(f^n(x), s\varepsilon)] \subset f^{n+1}(B(x, \varepsilon) \cap f^{-1}[B_{n-1}(f(x), s^2\varepsilon)]), \tag{15}$$

or

$$B(f^n(x), s\varepsilon) \subset f^n(B(x, \varepsilon) \cap f^{-1}[B_{n-1}(f(x), s^2\varepsilon)]), \tag{16}$$

where $B_k(x, \varepsilon)$ is defined in Definition 3.

Write

$$U_\varepsilon[C(f)] = \bigcup\{B(c, \varepsilon) | c \in C(f)\}.$$

By the choice of η and ε, this is a disjoint union of closed subintervals of size 2ε.

If $x \neq U_\varepsilon[C(f)]$, then (16) holds for $n(x) = 1$.

Given $c \in C(f)$, we check (15) or (16) for every $x \in B(c, \varepsilon)$.

Step 1. If $f(c) \in C(f)$, (15) holds for $n(x) = 1$.

Step 2. If $f(c) \notin C(f)$ but $f^2(c) \in C(f)$, then by the choice of ε,

$$B(p, s^4\varepsilon) \cap C(f) = \begin{cases} \{p\} & \text{for } p = c, f^2(c), \\ \varnothing & \text{for } p = f(c). \end{cases}$$

It is not hard to check the following (cf. [Ch]):

(1) when x satisfies either $\varepsilon/s \leq |x - c| \leq \varepsilon$ or $|x - c| < \varepsilon/s^2$, (15) holds for $n(x) = 2$;

(2) when x satisfies $\varepsilon/s^2 < |x - c| \leq \varepsilon/s$, (16) holds for $n(x) = 3$;

Step 3. If $f(c)$, $f^2(c) \notin C(f)$, choose ε small enough that for all $x \in B(c, \varepsilon)$ we have $[B(f(x), \varepsilon) \cup B(f^2(x), \varepsilon)] \cap C(f) = \varnothing$ and for every $a \in f^{-1}[C(f)] - C(f)$, $B(a, s^2\varepsilon) \cap C(f) = \varnothing$. Since $C(f)$ is linked by f, the smallest integer M for which $D_M(c, \varepsilon) \cap C(f) \neq \varnothing$ exists and satisfies $M \geq 3$.

Notice that for every $x \in B(c, \varepsilon)$, if k is the least k for which $D_k(x, \varepsilon) \cap C(f) \neq \varnothing$, then $D_{k-1}(x, s\varepsilon) \cap C(f) = \varnothing$.

Now we consider two cases:

Case 1. If $x \in B(c, \varepsilon/s^M)$, then by Lemma 13, the smallest k for which $D_k(x, \varepsilon) \cap C(f) \neq \varnothing$ satisfies $3 \leq k \leq M$. It follows from Lemma 12 that (15) holds for $n(x) = k$ and $N = M$.

Case 2. If $|x - c| > \varepsilon/s^M$, then for some $1 \leq L_x \leq M$, $\varepsilon/s^{L_x} < |x - c| \leq \varepsilon/s^{L_x - 1}$. Let

$$E_1 = \{x \in B(c, \varepsilon) | D_k(x, \varepsilon) \cap C(f) \neq \varnothing \text{ for some } k,\ 1 \leq k \leq L_x\}$$

and

$$E_2 = \{x \in B(c, \varepsilon) | D_k(x, \varepsilon) \cap C(f) = \varnothing \text{ for } j = 1, \dots, L_x\}.$$

First consider $x \in E_1$; (15) holds for $n(x) = k$ and $N = M$ since $L_x \leq M$.

Second, by Lemma 14, for $x \in E_2$ we have $D_{L_x}(x, \varepsilon) = B(f^{L_x}(x), \varepsilon)$. Thus, (16) holds for $n(x) = L_x + 1$ and $N = M + 1$;

Because $C(f)$ is finite, we can find a finite number M_c for each $c \in C(f)$ such that

$$D_{M_c}(c, \varepsilon) \cap C(f) \neq \varnothing .$$

Let $N = \max\{M_c + 1 | c \in C(f)\}$, then (15) or (16) holds for all $x \in I$ with $n(x) \leq N$. $\square$

4. Examples

1. *Tent maps.* The family of *tent maps* T_s: $[0, 2] \to [0, 2]$, $1 < s \leq 2$, is defined by

$$T_s(x) = \begin{cases} sx & \text{if } x \leq 1, \\ s(2-x) & \text{if } x > 1. \end{cases}$$

If $s \neq 2$, then $T_s^j(1) \in [T_s^2(1), T_s(1)]$ for all $j \geq 0$, where $[T_s^2(1), T_s(1)]$ is a proper subinterval of $(0, 2)$. So, 1 is linked to neither 0 nor 2 by T_s. Thus, if 1 is linked to 0 or 2, we must have $s = 2$. This implies that T_s has the Shadowing Property if and only if either $s = 2$ or 1 is linked to itself.

The condition formulated by Coven-Kan-Yorke for the Shadowing Property for a tent map T_s $(s \neq 2)$ was: for any $\varepsilon > 0$, there exists an integer $M \geq 1$ such that

$$1 \in \{f^m(y) | |f^j(y) - f^j(1)| \leq \varepsilon \text{ for } j = 0, \dots, M\}.$$

It is easy to see that in our language this is just the condition that 1 is linked to itself. Thus our main theorem generalizes Theorem 4.2 of [CKY].

2. *Transitive piecewise monotone maps of the interval.* A continuous map f of a compact metric space X to itself is (*topologically*) *transitive* if there exists a dense orbit of f in X. f is *strongly transitive* if for any open subset $U \subset X$, there is an integer $n \geq 0$ such that $\bigcup_{j=0}^n f^j(U) = X$ and f is *topologically mixing* if for every pair U, V of nonempty open sets, there exists an integer N such that $f^{-n}(U) \cap V \neq \varnothing$ for every $n \geq N$. It is easy to see that topologically mixing gives a dense orbit.

By Coven–mulvey [CM] (cf. Block-Coven [BC]), transitivity and strong transitivity are equivalent for a continuous piecewise monotone map $f: I \to I$.

In [P], W. Parry proved that every continuous strongly transitive piecewise monotone map of a compact interval is conjugate to a continuous uniformly piecewise linear map of the unit interval onto itself. This implies that

For a (strongly) transitive continuous piecewise monotone map of the interval, the Shadowing Property and the Linking Property are equivalent.

3. *Exact and semiexact piecewise monotone maps.* Suppose $f: I \to I$ is a map of a compact interval to itself. f is (*topologically*) *exact* if for each nontrivial interval $J \subset I$ there exists $n \geq 0$ such that $f^n(J) = I$, and f is *semiexact* if there exist closed intervals I_1, $I_2 \subset I$ with $I = I_1 \cup I_2$ and $\operatorname{Int} I_1 \cap \operatorname{Int} I_2 = \varnothing$ such that $f(I_1) = I_2$ and $f(I_2) = I_1$, and the restriction of

f^2 to I_1 is exact. It is easy to see that if f is exact, then it is topologically mixing; and if f is semiexact, then it is strongly transitive.

If f is a semiexact continuous piecewise monotone map with $I_1 = [a, c]$ and $I_2 = [c, b]$ where c is a fixed point of f, we claim there exists a turning point u such that $f(u) = c$. It follows from Example 2 that f does not have the Shadowing Property since c is a fixed point that is not in $C(f)$. To show the claim, let

$$m = \min\{f(u) | u \in C(f) \cap [a, c)\}$$

and

$$M = \max\{f(u) | u \in C(f) \cap (c, b]\}.$$

If either m or M equals c, then we are done; otherwise, set $A = f^{-1}([m, b])$ and $B = f^{-1}([a, M])$, then A and B are subintervals of I with $a \in A$ and $b \in B$ and $C(f) \subset A \cup B$. By the result of [P] referred to above, f is conjugate to a uniformly piecewise linear map, hence the image of c under this conjugacy is a repellor, and so is c. We can choose an $\varepsilon > 0$ small enough that $[c - \varepsilon, c + \varepsilon] \subset I - (A \cup B)$ and $[c - \varepsilon, c + \varepsilon] \subset f([c - \varepsilon, c + \varepsilon])$. Thus, $I - [c - \varepsilon, c + \varepsilon]$ is invariant for f, contradicting the (strong) transitivity of f.

In general, by a result in [Pr], if f is a continuous (strongly) transitive piecewise monotone map on I, then f is either exact or semiexact; but, it follows from the above that if f is semiexact, then f does not have the Shadowing Property. Hence

Corollary. *For a continuous piecewise monotone map f of a compact interval to itself with the Shadowing Property, the following statements are equivalent:*

(1) *f is (strongly) transitive;*
(2) *f is exact;*
(3) *f is topologically mixing.*

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