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ON THE COMMUTATIVITY OF ULTRAPRODUCTS WITH DIRECT PRODUCTS

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ABSTRACT. We show that an ultraproduct of direct products of structures is elementarily equivalent to a direct product (naturally defined over an ultraproduct of sets!) of ultraproducts of these structures.

It is a well-known fact of model theory that "finite products commute with ultraproducts": if J is a finite set, D is an ultrafilter on a set I, and $\{\mathfrak{A}_{i,j} \mid i \in I, j \in J\}$ is a set of structures, then there is a canonical isomorphism:

$$\Pi_D(\Pi_J \mathfrak{A}_{i,j}) \cong \Pi_J(\Pi_D \mathfrak{A}_{i,j})$$

(where $\Pi_D \mathfrak{B}_i$ is the D-ultraproduct of the $\mathfrak{B}_i$'s and $\Pi_J \mathfrak{B}_j$ is the direct product of the $\mathfrak{B}_j$'s; a more precise notation would be $\Pi_D(\Pi_{j\in J}\mathfrak{A}_{i,j};\ i \in I) \cong \Pi_{j\in J}(\Pi_D(\mathfrak{A}_{i,j};\ i \in I))$, but the context prevents any ambiguity). This can be checked directly (the isomorphism being the obviously defined one), or seen as a consequence of a very general algebraic property; namely, that "filtered colimits commute with finite limits" in a wide class of "natural" categories.

One readily finds counterexamples for infinite sets J. In fact, the commutativity cannot be extended to infinite J even if the isomorphism is replaced by an elementary equivalence: let $I = J = \mathbb{N}$, and consider a statement ϕ, which is true in a direct product $\Pi_K \mathfrak{B}_k$ if and only if it is true in each $\mathfrak{B}_k$ (for example $\phi = \forall x(R(x))$ for some relation symbol R), and a filter D on I containing all the cofinite subsets; then, if one chooses the structures $\mathfrak{A}_{i,j}$ such that $\mathfrak{A}_{i,j} \models \phi$ if and only if $i \geq j$, one gets $\Pi_J(\Pi_D \mathfrak{A}_{i,j}) \models \phi$ but $\Pi_D(\Pi_J \mathfrak{A}_{i,j}) \models \sim \phi$.

However, it seems to have been unnoticed before that there is a natural way to generalize to infinite sets J the elementary equivalence above. One has to take on the right hand side the ultrapower (of sets) $\Pi_D J$ instead of J (this really generalizes the finite case, as the natural embedding $J \to \Pi_D J$ an isomorphism when J is finite): more explicitly, for each $x \in \Pi_I J = J^I$, denote by $[x]_D$ its equivalence class in $\Pi_D J$; then we will see that

$$\Pi_D(\Pi_J \mathfrak{A}_{i,j}) \equiv \Pi_{[x]_D \in \Pi_D J}(\Pi_D \mathfrak{A}_{i,x(i)}).$$

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(One could find examples showing that replacing " $[x]_D \in \Pi_D J$ " by " $x \in \Pi_I J$ " invalidates the equivalence in general.) Furthermore, there is no necessity to keep J fixed, and we show the following more general fact:

Theorem 1. *Let D be an ultrafilter on a set I, $\{J_i\}_{i\in I}$ be a family of disjoint sets, and $\{\mathfrak{U}_j \mid j \in \bigcup_{i\in I} J_i\}$ be a set of structures. Then*

$$\Pi_D(\Pi_{J_i}\mathfrak{U}_j) \equiv \Pi_{[x]_D\in\Pi_D J_i}(\Pi_D \mathfrak{U}_{x(i)}).$$

(As above, this is a short notation for

$$\Pi_D(\Pi_{j\in J_i}\mathfrak{U}_j\,;\ i \in I) \equiv \Pi_{[x]_D\in\Pi_D J_i}(\Pi_D(\mathfrak{U}_{x(i)}\,;\ i \in I)).)$$

The essence of the proof lies on a result due to Vaught ([5]; see also [2] for more details). Here is its formulation in [1] (where one can also find a proof):

Theorem 2. *Given a sentence ϕ, we can (effectively) find a number n such that for all index sets I and all structures $\mathfrak{U}_i$, $i \in I$, there is a subset J of I with at most n elements such that for all K, $J \subset K \subset I$,*

$$\Pi_{i\in I}\mathfrak{U}_i \models \phi \quad \textit{if and only if} \quad \Pi_{i\in K}\mathfrak{U}_i \models \phi. \quad \square$$

Proof of Theorem 1. (a) We first assume that all cardinalities $|J_i|$ are infinite.

Let ϕ be a sentence such that $\Pi_D(\Pi_{J_i}\mathfrak{U}_j) \models \phi$, i.e. $\{i \in I \mid \Pi_{J_i}\mathfrak{U}_j \models \phi\} \in D$. By Theorem 2, there exist $m = m_\phi \in \mathbb{N}$ and $\{J_i'\}_I$ where $J_i' \subset J_i$, $|J_i'| \le m$ for each $i \in I$, and such that for every $\{J_i''\}_I$ with $J_i' \subset J_i'' \subset J_i$, $i \in I$, we have $\Pi_{j\in J_i}\mathfrak{U}_j \models \phi$ if and only if $\Pi_{j\in J_i''}\mathfrak{U}_j \models \phi$.

Because all $|J_i|$ are infinite, we can find $\{J_i''\}_{i\in I}$ such that $|J_i''|$ is precisely m for each $i \in I$, and such that $\{i \in I \mid \Pi_{J_i''}\mathfrak{U}_j \models \phi\} \in D$, i.e. $\Pi_D(\Pi_{J_i''}\mathfrak{U}_j) \models \phi$.

To clarify, we write $\Pi_{J_i''}\mathfrak{U}_j$ explicitly as $\mathfrak{U}_{j_1(i)} \times \cdots \times \mathfrak{U}_{j_m(i)}$. For each $k \in \{1, \ldots, m\}$, let x_k be the element of $\Pi_I J_i$ such that $x_k(i) = j_k(i)$ for every $i \in I$. Because m is finite, we get

$$\Pi_D(\Pi_{J_i''}\mathfrak{U}_j) \cong \Pi_D(\mathfrak{U}_{x_1(i)}) \times \cdots \times \Pi_D(\mathfrak{U}_{x_m(i)}).$$

Let $x' \in \Pi_I J_i$ be such that $[x']_D \ne [x_k]_D$ for $k = 1, 2, \ldots, m$. Because all $|J_i|$ are infinite, we can suppose that $x'(i) \ne x_k(i)$ for each $i \in I$ and each $k \in \{1, 2, \ldots, m\}$. Now we have

$$\Pi_D(\mathfrak{U}_{x_1(i)}) \times \cdots \times \Pi_D(\mathfrak{U}_{x_m(i)}) \times \Pi_D(\mathfrak{U}_{x'(i)}) \cong \Pi_D(\mathfrak{U}_{x_1(i)} \times \cdots \times \mathfrak{U}_{x_m(i)} \times \mathfrak{U}_{x'(i)}).$$

As for each $i \in I$, $\{x_1(i), \ldots, x_m(i), x'(i)\}$ is a set of distinct elements of J_i containing J_i'', one can see that $\Pi_D(\mathfrak{U}_{x_1(i)} \times \cdots \times \mathfrak{U}_{x_m(i)} \times \mathfrak{U}_{x'(i)}) \models \phi$.

We can repeat the argument for any finite number of x', hence we have shown that:

(+) For every finite set $\{\Pi_D, (\mathfrak{U}_{x_1'(i)}), \ldots, \Pi_D(\mathfrak{U}_{x_n'(i)})\}$ of ultraproducts such that the $[x_1']_D, \ldots, [x_n']_D$ are distinct and distinct from each one of $[x_1]_D, \ldots, [x_m]_D$, we have $\Pi_D(\mathfrak{U}_{x_1(i)}) \times \cdots \times \Pi_D(\mathfrak{U}_{x_m(i)}) \times \Pi_D(\mathfrak{U}_{x_1'(i)}) \times \cdots \times \Pi_D(\mathfrak{U}_{x_n'(i)}) \models \phi$.

If we now suppose that $\Pi_Y(\Pi_D(\mathfrak{A}_{x(i)})) \models\sim \phi$, for $Y = \Pi_D J_i$, then, by Theorem 2 again, there exists a finite subset Y' of Y such that for every $Y'' \subset Y$ disjoint from Y', we have $\Pi_{Y''}(\Pi_D(\mathfrak{A}_{x(i)})) \times \Pi_{Y'}(\Pi_D(\mathfrak{A}_{x(i)})) \models\sim \phi$. This contradicts (+) (take $Y'' = \{[x_1]_D, \ldots, [x_m]_D\}$).

(b) For the general case, let $\alpha = \sup_{i\in I}(|J_i|, \omega)$. We add to each set $\{\mathfrak{A}_j \mid j \in J_i\}$ a sufficient number of "trivial" structures $\{*\}$ to get sets J_i' of cardinality α. (A trivial structure is one with only one element $*$ and such that $R(*, *, \ldots, *)$ for every relation R.) Let us denote these new sets by $\{\mathfrak{A}'_{j'} \mid j' \in J_i'\}$, where $\mathfrak{A}'_{j'} = \{*\}$ if $j' \in J_i' \setminus J_i$ and $\mathfrak{A}'_{j'} = \mathfrak{A}_{j'}$ otherwise. Clearly $\Pi_D(\Pi_{J_i}\mathfrak{A}_j) \cong \Pi_D(\Pi_{J_i'}\mathfrak{A}'_{j'})$.

Let us consider an ultraproduct $\Pi_D(\mathfrak{A}'_{x'(i)})$, where $x'(i) \in \Pi_I J_i'$. We distinguish two cases:

(1) If $\{i \mid x'(i) \in J_i\} \in D$, then $\Pi_D(\mathfrak{A}'_{x'(i)}) \cong \Pi_D(\mathfrak{A}_{x(i)})$, where $x(i) = x'(i)$ if $x'(i) \in J_i$ and $x(i)$ is any element of J_i if $x'(i) \in J_i' \setminus J_i$.

(2) If $\{i \mid x'(i) \in J_i\} \notin D$, then $\Pi_D(\mathfrak{A}'_{x'(i)}) \cong \{*\}$.

From (1) and (2), we deduce that $\Pi_{\Pi_D J_i'}(\Pi_D \mathfrak{A}'_{x'(i)}) \cong \Pi_{\Pi_D J_i}(\Pi_D \mathfrak{A}_{x(i)})$. By part (a), we have $\Pi_{\Pi_D J_i'}(\Pi_D \mathfrak{A}'_{x'(i)}) \equiv \Pi_D(\Pi_{J_i'}\mathfrak{A}'_j)$, and then the result. □

As a corollary, one easily obtains the main result of [4], which states that if K is a *compact* class of structures (i.e. $\{\mathfrak{A} \mid \mathfrak{A} \equiv \mathfrak{B}$ for some $\mathfrak{B} \in K\}$ is elementary), then $\mathbf{P}(K)$ (the class of all direct products of structures in K) is also compact. (The proof of Makkai, which is quite complicated, lies on a much refined version of Theorem 2 also found in [2]). This has interesting consequences on the strength of the preservation theorem for direct products (see [3]).

The referee pointed out to us that Theorem 1 follows more easily from an unpublished work of Y. Vourtsanis [6]. As in the refined version of Theorem 2 mentioned above (but in a different way), the main result of [6] associates to a given formula ϕ a finite set $\{\phi_k\}$ of formulas and then nicely characterizes the satisfaction of ϕ in a direct product in terms of the satisfaction of the ϕ_k's in the factor.

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