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THE LOXODROMIC TERM OF THE SELBERG TRACE FORMULA FOR $SL(3, \mathbb{Z}) \setminus SL(3, \mathbb{R})/ SO(3, \mathbb{R})$

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ABSTRACT. In this paper we calculate the contribution to the trace formula of those orbital integrals coming from matrices in $SL(3, \mathbb{Z})$ with two complex eigenvalues and one real one, none of which are equal to zero. These correspond to mixed cubic number fields and will be seen to occur with multiplicity equal to the class number of a certain order in the number field.

The purpose of this paper is to compute the term of the Selberg trace formula for $SL(3, \mathbb{Z})$ that comes from a certain sum of orbital integrals for an $SO(3, \mathbb{R})$ bi-variant kernel, ϕ, where the orbit is taken over elements of $SL(3, \mathbb{Z})$ that have two complex and one real eigenvalue, none of which are equal to ± 1. In this paper we refer to this term as the loxodromic term of the trace formula (as does Efrat, in [1]) and elements of this type are called loxodromic elements. For $SL(4, \mathbb{Z})$ and higher dimensional groups, it is not a very apt designation since there are many more kinds of terms that, for the purposes of computing anything, must be distinguished from each other.

In [6] we have given a description of the trace formula for $SL(3, \mathbb{Z})$ acting on $SL(3, \mathbb{R})/ SO(3, \mathbb{R})$ in connection with which the loxodromic orbital integrals arise, so we skip all this background information and let the reader read it there. We begin with the fact that in the trace formula the following term arises and must be evaluated:

$$(1) \qquad \int_{Y\in\mathbb{F}} \sum_{\substack{\gamma\in\Gamma \\ \text{loxodromic}}} \phi(Y^{-1}\gamma Y)\, d\mu(Y).$$

Here, Y is an element of the coset space $SL(3, \mathbb{R}) \setminus SO(3, \mathbb{R})$ and ϕ is an $SO(3, \mathbb{R})$ bi-invariant function that is C^∞ with compact support. Because of the $SO(3, \mathbb{R})$ invariance, it does not matter which coset representative of

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Y and Y^{-1} we choose. Using the Iwasawa decomposition, we can write $\widetilde{Y} \in \mathrm{SL}(3, \mathbb{R})$ as

$$(2) \qquad \widetilde{Y} = \begin{pmatrix} y_1 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & y_1^{-2} \end{pmatrix} \begin{pmatrix} u^{+1/2} & v_1 u_1^{-1/2} & 0 \\ 0 & u_1^{-1/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & t_1 \\ 0 & 0 & 1 \end{pmatrix} k,$$

where $k \in \mathrm{SO}(3, \mathbb{R})$. So we can use $\{(y_1, u_1, v_1, x_1, t_1)|_{v_1, x_1, t_1, |\mathbb{R}}^{y_1, u_1 > 0}\}$ as coordinates for Y. Y^{-1} is of course given by

$$Y^{-1} = \begin{pmatrix} 1 & 0 & -x_1 \\ 0 & 1 & -t_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1^{-1/2} & -v_1 u_1^{-1/2} & 0 \\ 0 & u_1^{1/2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1^{-1} & 0 & 0 \\ 0 & y_1^{-1} & 0 \\ 0 & 0 & y_1^2 \end{pmatrix}.$$

The measure $d_\mu(Y)$ is an $\mathrm{SL}(2, \mathbb{R})$ invariant measure on the coset space normalized so that $\int_{\mathrm{SO}(3, \mathbb{R})} \phi(kY)\, d\mu(k) = \phi(Y)$.

The region $\mathbb{F}$ is the fundamental region for the action of $\Gamma = \mathrm{SL}(3, \mathbb{Z})$, which is described somewhat by Venkov [4] and Grenier [2]. Its essential features are that it can be contained in a half cylinder, $\{y | y_1 > c_1,\ u_1 > c_2,\ |v_1|, |x_1|, |t_1| \leq 1/2\}$ and that it has a two-dimensional cusp at $y_1 = \infty$, $u_1 = \infty$. For this computation, we do not even need that much information.

In [5] we showed that the hyperbolic elements of $\mathrm{SL}(N, \mathbb{Z})$ occurred in conjugacy classes according to their eigenvalues and that the number of these classes equals the narrow class number of the order generated by the eigenvalues. The hyperbolic elements are just the case where the eigenvalues are all real. Olga Todd immediately pointed out that the results extend to the case of mixed eigenvalues (not necessarily real), none of which are equal to each other or to ± 1. So we can split the sum in (1) into a double sum: (c denotes class number)

$$\sum_{\substack{\gamma \in \Gamma \\ \Gamma \text{ loxodromic}}} = \sum_{\lambda = \{\lambda_1, \lambda_2, \lambda_3\}} c(O_\lambda) \sum_{\gamma \in \{\gamma_\lambda\}},$$

where $\{\lambda_1, \lambda_2, \lambda_3\}$ are eigenvalues generating an order 0_λ in some field K and $\{\gamma_\lambda\}$ is a conjugacy class corresponding to λ.

Now $\{\lambda_1, \lambda_2, \lambda_3\}$ are units of positive norm in the cubic number field K. The group of units of positive norm in K is a cyclic group on some generator corresponding to the set of eigenvalues $\lambda_0 = \{r_0 e^{i\theta_0}, r_0 e^{-i\theta_0}, r_0^{-2}\}$. Following the results in [5] and [6], we conclude that the centralizer of a loxodromic γ is cyclic with generator γ_0 that has eigenvalues λ_0. We can write (1) as

$$\begin{aligned} &\sum_{\lambda = \{\lambda_1, \lambda_2, \lambda_3\}} c(0_\lambda) \sum_{\lambda \in \{\gamma_\lambda\}} \int_{Y \in \mathbb{F}} \phi(Y^{-1} \gamma Y)\, d\mu(Y) \\ &= \sum_\lambda c(0_\lambda) \sum_{\sigma \in Z(\gamma) \backslash \mathrm{SL}(3, \mathbb{Z})} \int_{Y \in \mathbb{F}} \phi(Y^{-1} \sigma^{-1} \gamma \sigma Y)\, d\mu(Y), \end{aligned}$$

where $Z(\gamma)$ is the centralizer of γ,

$$(3) \qquad = \sum_{\lambda} \int_{\mathbb{F}_{\tilde{\gamma}}} f(Y^{-1}\tilde{\gamma}Y)\, d\mu(Y),$$

where $\mathbb{F}_\gamma$ is a choice of fundamental region for $Z(\gamma)$, $\tilde{\gamma}$ is some choice of conjugate in $\mathrm{SL}(3, \mathbb{R})$ of γ and $\mathbb{F}_{\tilde{\gamma}}$ is the image of $\mathbb{F}_\gamma$ under the chosen change of variables. This calculation is standard. For example, see [6].

Now γ is the element of some Cartan subgroup of $\mathrm{SL}(3, \mathbb{R})$ that is not split over $\mathbb{R}$. In other words, γ is conjugate under $\mathrm{SL}(3, \mathbb{R})$ to

$$(4) \qquad \tilde{\gamma} = \begin{pmatrix} rR_\theta & & 0 \\ & & 0 \\ 0 & 0 & r^{-2} \end{pmatrix}, \qquad \text{where } R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Furthermore, the generator of $Z(\gamma)$ goes, under the same change of variables, to

$$\tilde{\gamma}_0 = \begin{pmatrix} r_0 R_\theta & & 0 \\ & & 0 \\ 0 & 0 & r_0^{-2} \end{pmatrix}.$$

A fundamental region for the action of $\tilde{\gamma}_0$ in the coordinates given earlier is $\mathbb{F}_{\tilde{\gamma}} = \{Y | 1 \le y_1 < r_0,\ u_1 > 0,\ x_1,\ t_1,\ v_1 \in \mathbb{R}\}$.

Writing Y and $\tilde{\gamma}$ as in (2) and (4) and doing some arithmetic to the formula in (3) yields

(5)

$$\sum_{\lambda} c(0_\lambda) \int_{\widetilde{Y} \in \mathbb{F}_{\tilde{\gamma}}} \phi\left(\begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r^{-2} \end{pmatrix} \begin{pmatrix} X^{-1}R_\theta X & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & x^* \\ 0 & 1 & t^* \\ 0 & 0 & 1 \end{pmatrix} \right) d\mu(Y),$$

where

$$X = \begin{pmatrix} u_1^{-1/2} & u_1^{-1/2} v_1 \\ 0 & u_1^{-1/2} \end{pmatrix}$$

and

$$\begin{pmatrix} x^* \\ t^* \end{pmatrix} = (I_{2x2} - r^{-3}(X^{-1}R_\theta X)) \begin{pmatrix} x_1 \\ t_1 \end{pmatrix}.$$

In these coordinates, $d\mu(Y) = dx\, dt\, d\mu (du/u^2)(dy/y)$. The determinant of the Jacobian of this transformation is given by

$$|J| = |1 - r^{-3}(2\cos\theta) + r^{-6}|.$$

Making the change of variables and integrating out y_1 leaves (5) in the form

$$(6) \quad \sum_{\lambda} \frac{c(0_\lambda)|\log(r_0)|}{|1 - 2r^{-3}\cos\theta + r^{-6}|}$$

$$\times \int_{u>0} \iiint_{v,x,t \in \mathbb{R}^3} \phi\left(\begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r^{-2} \end{pmatrix} \begin{pmatrix} X^{-1}R_\theta X & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \right) dx\, dt \frac{dy\, dv}{u^2}.$$

Here we have written out the remaining part of $d\mu(Y)$ and we note that if we regard X as the point in the Poincaré upper halfplane, H, given by $v + iu$, then $(1/u^2)\,dv\,du$ is the $SL(2, \mathbb{R})$ invariant measure associated to $\mathbb{H}$ in the usual way.

The transform given by

$$(7)\qquad \hat{\phi}(Y) = \int_{x\in\mathbb{R}} \int_{t\in\mathbb{R}} \phi\left(Y \cdot \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}\right) dx\,dt$$

is a transform that occurs other places in the Selberg trace formula for this group and is variously called an Harish, Harish–Chandra, or Abel transform. (Other similar transforms are also called by all these names.) The integrand in (6) is equal to

$$\hat{\phi}\left(\begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r^{-2} \end{pmatrix} \begin{pmatrix} X^{-1}R_\theta X & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0\ \ 0 & 1 \end{pmatrix}\right)$$

after integrating out x and t. $\hat{\phi}$ is $SO(2, \mathbb{R})$ bi-invariant in the coordinates $v + iu \in \mathbb{H}$ with the usual action.

A slight modification of the argument in [7, p. 12] tells us that

(8)

$$\hat{\phi}\left(\begin{pmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r^{-2} \end{pmatrix} \begin{pmatrix} X^{-1}R_\theta X & \\ & 1 \end{pmatrix}\right) u^t \frac{du\,dv}{u^2} = \left(\int_{\mathrm{Re}\, s=1/2} h_\theta(s, t) r^{1-s}\,ds\right) u^t,$$

where h_θ is the Selberg transform of ϕ_θ and where

$$\phi_\theta(r, u, v, x, t) = \phi(r, u', v', x, t)$$

$$u' = \mathrm{Im}((X^{-1}R_\theta X) \circ i), \qquad X = \begin{pmatrix} \sqrt{v} & v/\sqrt{u} \\ 0 & 1\sqrt{u} \end{pmatrix}$$

$$v' = \mathrm{Re}((X^{-1}R_\theta X) \circ i).$$

Equation (9) says that $\int_{\mathrm{Re}\, s=1/2} h_\theta(s, t) r^{1-s}\,ds$ is the $SL(2, \mathbb{R})$ Selberg transform of $\hat{\phi}_\theta(r, u, v)$ where r is considered fixed. Now if we set

$$\check{h}(t) = \int_{\mathrm{Re}\, s=1/2} h(s, t) r^{1-s}\,dx$$

and $\tilde{\phi}_\theta(u, v) = \hat{\phi}_\theta(r, u, v)$, a calculation due to Selberg and found in Kubota, [3], tells us how to write $\tilde{\phi}_\theta$ in terms of $\check{h}$. Using it, we obtain

$$(9)\qquad \int_{iu+v\in\mathbb{H}} \tilde{\phi}_\theta(u, v) \frac{du\,dv}{u^2} = \frac{1}{\sin\theta} \int_{\mathrm{Re}\, t=1/2} \frac{e^{-2\theta t}}{1 + e^{-2\pi t}} \check{h}(t)\,dt.$$

This tells us that the loxodromic orbital integral in (9) is equal to

$$(10)\qquad \int_{\mathrm{Re}\, t=l/2} \int_{\mathrm{Re}\, s=1/2} h(s, t) \frac{r^{1-s} e^{-2\theta t}}{1 + e^{-2\pi t}}\,ds\,dt.$$

Using the formula plus the fact that we can write the pair (r, θ) as (r_0^k, θ_0^k) where (r_0, θ_0) generate the units of the number field, we have the following theorem:

Theorem. *With notation as before, the total contribution of the loxodromic term to the Selberg trace formula for* $\mathrm{SL}(3,\mathbb{Z})\backslash \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3,\mathbb{R})$ *is given by the term*

$$\sum_{\substack{K \\ \text{distinct} \\ \text{mixed cubic} \\ \text{number fields}}} \sum_{k>0} \frac{c(0_k)|\log r_0|}{|1-2r_0^{-3k}\cos\theta_0^k + r_0^{-6k}|}$$

$$\circ \int_{\mathrm{Re}\, t=1/2} \int_{\mathrm{Re}\, s=1/2} h(s,t) \frac{r_0^{k(1-s)} e^{-2\theta_0^k t}}{1+e^{-2\pi t}}\, ds\, dt .$$

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