PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 113, Number 2, October 1991

THE SECOND BIRKHOFF THEOREM FOR OPTICAL HAMILTONIAN SYSTEMS

LEONID POLTEROVICH

(Communicated by Kenneth R. Meyer)

ABSTRACT. Consider a smooth Hamiltonian function on the cotangent bundle of the n-dimensional torus such that its restriction on every fiber is strictly convex. Let L be a Lagrange invariant torus of the Hamiltonian flow which is homologous to the zero section. We show that, under some assumptions, L is a smooth section of the cotangent bundle.

1. INTRODUCTION AND MAIN RESULTS

In [B], G. D. Birkhoff developed the qualitative theory of symplectic twist maps of the cylinder $C = \mathbf{R}^1 \times \mathbf{S}^1$ (see [H1] for a modern survey). One of his theorems (which is known as the second Birkhoff theorem) states that any embedded noncontractible invariant circle of such a symplectomorphism is a section of the $\mathbf{R}$-bundle C over $\mathbf{S}^1$. In the present paper we prove the analogous assertion for Lagrange invariant tori of optical Hamiltonian systems on the cotangent bundle $T^*\mathbf{T}^n$ of the n-dimensional torus.

Definition. A smooth Hamiltonian function $H\colon T^*\mathbf{T}^n \to \mathbf{R}$ is called optical if its restriction to every fiber of the cotangent bundle is strictly convex (i.e., $H_{pp} > 0$, where p is a linear coordinate on a fiber).

Example. The Hamiltonian function associated with a Riemannian metric on the torus is optical.

Main Theorem. *Let $L \subset T^*\mathbf{T}^n$ be a smooth Lagrange torus which lies in a regular level $\{H = \text{const}\}$ of an optical Hamiltonian function. Assume that*

(c1) *L is homologous to the zero section;*
(c2) *the Hamiltonian flow on L preserves a smooth measure.*

Then L is a smooth section of the cotangent bundle.

Remarks.

1. For the case $n = 2$, this theorem (under slightly different assumptions) was proved together with M. Bialy in [BP1] (see also [Bi], [Po]). In the present work we use to a great extent the ideas of [BP1].

Received by the editors November 11, 1989.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 58F05; Secondary 53C57.

©1991 American Mathematical Society
0002-9939/91 \$1.00 + \$.25 per page

2. In [A1], V. I. Arnold suggested a program of studying topological properties of optical Lagrange submanifolds. This program includes the hypothesis that (under some additional assumptions) such a submanifold is a section of the cotangent bundle.

3. One can find some interesting generalizations of Birkhoff's theory to the n-dimensional case in an article by M. Herman [H2].

4. The following result is crucial for our proof:

Proposition [Vi], [LS]. *The Maslov class of any embedded Lagrange torus in* $T^*\mathbf{T}^n$ *which is homologous to the zero section is zero.*

The proof of this proposition is based on the result by C. Viterbo [Vi] concerning the Maslov class of embedded Lagrange tori in $\mathbf{C}^n$.

5. It seems that the assertion of the theorem holds without the measure-preserving condition (c2). However, this condition is rather natural for Hamiltonian systems of classical mechanics. For example, it holds for Lagrange invariant tori carrying quasi-periodic motion.

The following corollary is a simple generalization of [BP2, Theorem 1a], see also [BP1].

Corollary 1. *Consider a smooth Lagrange invariant torus of the Hamiltonian system associated with a Riemannian metric on* $\mathbf{T}^n$. *Assume that the conditions* (c1) *and* (c2) *of the main theorem hold. Then the natural projection of any trajectory of the Hamiltonian flow which lies on this torus is a minimal geodesic* (*this means that a lifting of such a geodesic to the covering space* $\mathbf{R}^n$ *minimizes the Riemannian distance between any two of its points*).

Corollary 2. *Let* d *be a Riemannian metric on* $\mathbf{T}^n$ *which has a "bump" in the following sense: There exists an embedded ball* $B^n \subset \mathbf{T}^n$ *and a point* $K \in B$ *such that* $2d(K, \partial B) > \operatorname{diam}(\partial B)$, *where* $\operatorname{diam}(\partial B)$ *is the diameter of* ∂B *in the induced Riemannian metric on* ∂B. *Then the geodesic flow associated with the metric* d *has no Lagrange invariant tori satisfying the conditions* (c1) *and* (c2) *of the main theorem.*

Proof of Corollary 2. Due to Corollary 1, it is sufficient to show that no minimal geodesic of the metric d passes through the point K. Suppose that such a minimal geodesic exists. Then it contains a segment g such that $K \in g \subset B$ and its ends lie on the boundary ∂B. Thus $2d(K, \partial B) \leq \operatorname{length}(g) \leq \operatorname{diam}(\partial B)$. This inequality contradicts our assumptions. □

Remark 6. An analogous example of a Riemannian metric with a "bump" on a two-dimensional torus was constructed in [BP1], [Ba], and [BR2].

2. Proof of the main theorem

1. Denote by $\pi: T^*\mathbf{T}^n \to \mathbf{T}^n$ the natural projection. Define the *critical set* $S(W)$ of a closed Lagrange submanifold W as the set of critical points of the

restriction $\pi|_W$. We will say that W is *generic* if S consists of an $(n-1)$-dimensional submanifold S^0 and its boundary $S \setminus S^0$ whose codimension in W is not less than 3.

2. We claim that it is sufficient to prove the theorem for Lagrange tori which are generic. It is well known [A2, Appendix 12] that there exists a sequence F_k of symplectomorphisms of the cotangent bundle such that $\{F_k\}$ C^∞-converges to the identity and all $F_k(L)$ are generic.

It is clear that, for large k, all Lagrange tori $F_k(L)$ satisfy the conditions of the theorem with the optical Hamiltonian function $H_k = H \circ F_k^{-1}$. Due to our assumption, each $F_k(L)$ is the graph of a smooth map $\Phi_k : \mathbf{T}^n \to \mathbf{R}^n$ (we identify $T^*\mathbf{T}^n$ with $\mathbf{T}^n \times \mathbf{R}^n$).

It is easy to check that the property of H_k of being optical implies that the corresponding Hamiltonian field is monotone positive in the sense of M. Herman [H2, p. 15]. Thus all the maps Φ_k satisfy the a priori Lipschitz estimate $\|D\Phi_k\|_{L_\infty} < C$, where C depends only on H and the norm of the perturbation $\|F_k - \mathrm{id}\|_{C^\infty}$. These estimates and the smoothness of L give us the result that L is the graph of some smooth map $\mathbf{T}^n \to \mathbf{R}^n$. Our claim is proved.

3. In the following we will assume that L is generic. Suppose that the critical set S is nonempty.

Lemma 1 [Ch]. *The Hamiltonian field v is not tangent to the critical set S.*

Thus the Hamiltonian field determines a coorientation (and the orientation) of the cycle S.

Lemma 2. *The cycle $[S]$ is dual to the Maslov class of the Lagrange torus L.*

This lemma is a simple generalization of Theorem 2.1a of [BP1]. To prove it, it is sufficient to check that the Hamiltonian field in the nonsingular points of S is directed to the positive side with respect to the canonical Maslov coorientation. The proof is based on the fact that H is optical.

Lemma 3. *The cycle S is homologous to zero*: $[S] = 0$.

Proof. The Maslov class of the Lagrange torus L is zero (see Remark 4). The statement of the lemma follows now from Lemma 2. □

Thus there exists an n-dimensional chain U such that $\partial U = S$. Denote by Ω the volume form which is preserved by the Hamiltonian field v. This means that $d i_v \Omega = 0$. From Lemma 1 one can conclude that $\int_S i_v \Omega \neq 0$. On the other hand, $\int_S i_v \Omega = \int_U d i_v \Omega = 0$. This contradiction proves the theorem. □

Acknowledgments

I am deeply grateful to V. I. Arnold for useful discussions and to J. -C. Sikorav who acquainted me with the result mentioned in Remark 4.

REFERENCES

[A1] V. I. Arnold, *First steps in symplectic topology*, Russian Math. Surv. **41** (1986), 1–21.

[A2] ——, *Mathematical methods of classical mechanics*, Berlin, Heidelberg, New York, 1978.

[B] G. D. Birkhoff, *Surface transformations and their dynamical applications*, Acta Math. **43** (1920), 1–119; Collected Math. Papers, vol. 2, pp. 119–229.

[Ba] V. Bangert, *Mather sets for twist maps and geodesics on tori*, Dynamics Reported **1** (1987),

[Bi] M. L. Bialy, *Aubry-Mather sets and Birkhoff's theorem for geodesic flows on the two-dimensional torus*, Comm. Math. Phys. **126** (1989), 13–24.

[BP1] M. L. Bialy and L. V. Polterovich, *Lagrangian singularities of invariant tori of Hamiltonian systems with two degrees of freedom*, Invent. Math. **97** (1989), 291–303.

[BP2] ——, *Geodesic flows on the two-dimensional torus and phase transitions "commensurability-incommensurability,"* Funct. Anal. Appl. **20** (1986), 260–266,

[Ch] Yu. V. Chekanov, *Caustics in geometric optics*, Funct. Anal. Appl. **20** (1986), 223–226.

[H1] M. Herman, *Sur les courbes invariantes par les diffeomorphismes de l'anneau*, Asterisque **103–104** (1983).

[H2] ——, *Existence et non existence de tores invariants par des diffeomorphismes symplectiques*, Ecole Polytechnique, Centre de Mathematiques, Equations aux derivees partielles (Seminaire 1987–1988), Expose **14** (1988).

[LS] F. Lalonde and J.-C. Sikorav, *Sous-varietes lagrangiennes exactes des fibres cotangents*, Comment. Math. Helv. (to appear).

[Po] L. V. Polterovich, *The Maslov class of the Lagrange surfaces and Gromov's pseudo-holomorphic curves*, Trans Amer. Math. Soc. (to appear).

[Vi] C. Viterbo, *A new obstruction to embedding Lagrangian tori*, Invent. Math. **100** (1990), 301–320.

107392, HALTURINSKAJA 4-2-60, MOSCOW, USSR AFFILIATION

Current address: School of Mathematical Sciences, Tel-Aviv University, Ramat-Aviv, 69978 Tel-Aviv, Israel