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PRIME FACTORS OF CONJUGACY CLASSES OF FINITE SOLVABLE GROUPS

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ABSTRACT. A bound for the number of primes dividing $[G : Z(G)]$ for certain finite solvable groups G is given in terms of the maximal number of primes dividing a conjugacy class.

INTRODUCTION

All groups considered in this paper are finite. If A is a nonempty finite set, let $|A|$ denote the number of distinct elements in A. If n is a positive integer and $\prod_{i=1}^{k} p_i^{a_i} = n$ is the factorization of n into distinct prime powers, let $w(n) = k$ and $\pi(n) = \{p_1, \dots, p_k\}$. If $x \in G$, $h_x = |G|/|C_G(x)|$ denotes the number of elements in the conjugacy class of x. For a group G, let $\alpha(G) = \max\{w(h_x) \mid x \in G\}$ and $\beta(G) = \{p \mid p$ a prime and $p \mid h_g$ for some $g \in G\}$. Let $\sigma(G) = \max\{w(\chi(1)) \mid \chi$ an irreducible character of $G\}$, and $\rho(G) = \{p \mid p$ a prime and $p \mid \chi(1)$ for some irreducible character χ of $G\}$.

It is well known that if G is a finite solvable group then $p \mid h_x$ for some $x \in G$ if and only if $p \mid |G/Z(G)|$.

Huppert has conjectured that if G is a finite solvable group, then $|\rho(G)| \leq 2\sigma(G)$. This conjecture has been verified when $\sigma(G) \leq 2$ (see [6, 2]). Gluck [2] has also shown $|\rho(G)| \leq 2\sigma(G)$ if all irreducible characters have squarefree degree.

Recently there have been theorems showing a parallelism between results for characters and results for conjugacy classes. At the 1989 International Group Theory Conference in Bressanone, Professor Huppert asked whether this parallelism extended to relating $\beta(G)$ and $\alpha(G)$ in a way analogous to results relating $\rho(G)$ and $\sigma(G)$. There is some indication that this is possible since $|\beta(G)| \leq 2\alpha(G)$ if $\alpha(G) = 1$ [1]. We note that $|\beta(G)| \leq 2\alpha(G)$ is the best possible bound in the sense that for each m, there is a group G_m with $|\beta(G_m)| = 2m$ and $\alpha(G_m) = m$. In this paper we find additional parallels to the work of Gluck and the conjecture of Huppert. The following theorems are proved.

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Theorem A. *Assume G is a finite supersolvable group and G/G_1 is cyclic where G_1 is the intersection of all maximal normal subgroups of G, then $|\beta(G)| \leq 2\alpha(G)$.*

Theorem B. *Assume G is a finite supersolvable group and $p \geq \alpha(G)$ for all prime divisors p of G, then $|\beta(G)| \leq 2\alpha(G)$.*

Corollary C is obvious once Theorem B is proved.

Corollary C. *Assume G is a supersolvable group of odd order and h_x is divisible by at most n distinct primes for all $x \in G$ where $n \leq 3$, then $|G/Z(G)|$ has at most $2n$ distinct prime divisors.*

Again, paralleling the work of Gluck [2], Theorem D is proved.

Theorem D. *Assume G is a finite solvable group such that h_x is squarefree for all $x \in G$, then $|\beta(G)| \leq 2\alpha(G)$.*

1

Throughout this section, if K is a normal subgroup of G and A a subset of G, $\overline{A}$ denotes the image of A in $G/K = \overline{G}$. The following elementary lemma is included for completeness.

Lemma 1. *Assume G is a finite solvable group.*

(i) *If $G = ST$ where $T \Delta G$, S is abelian, T is nilpotent, $(|S|, |T|) = 1$ and $C_S(T) = 1$, then there is a $y \in T$ and $|S| \mid h_y$.*

(ii) *If $M \Delta G$, then $|M/C_M(x)| \mid |G/C_G(x)|$ for $x \in M^\#$.*

Proof. (i) Let G be a minimal counterexample. Since $(|S|, |T|) = 1$, the same hypothesis holds for $\overline{G} = ST/[T, T]$ and we can assume T is abelian. Let T_1 be a maximal proper normal subgroup of G lying in T. If $T_1 \neq 1$, let $S_1 = C_S(T/T_1)$. Since $|G/T_1| < |G|$, the minimality of G implies that $|S_1| \neq 1$. Now T/T_1 a chief factor and $(|S|, |T|) = 1$ yield $T = C_T(S_1)T_1$. Hence, $S_1T_1 \Delta G$. Let $\overline{G} = G/S_1T_1$, then there is an $x \in C_T(S_1)$ such that $|\overline{S}| \mid h_{\bar{x}} = |\overline{G}/C_{\overline{G}}(\bar{x})|$. Since $C_S(T) = 1$ and S_1 normalizes T_1, $C_{S_1}(T_1) = 1$. Again $|G| > |S_1T_1|$ implies there is a $z \in T_1$ with $|S_1| \mid |S_1T_1|/|C_{S_1T_1}(z)|$. Let $y = xz$, if $|S| \nmid h_y$, then T abelian implies there is a $g \in S$ with $y^g = y$. Since g centralizes y, g centralizes x (mod T_1). Now T/T_1 a chief factor of G and S abelian imply that g centralizes T/T_1 whence $g \in S_1$ and g centralizes x. Now $xz = y = y^g = xz^g$ yields $z^g = z$ so that $g = 1$.

Hence, $T_1 = 1$ and T is an elementary p-group. Since S is also abelian, $C_T(x) \Delta G$ for all $x \in S$. Hence, $C_S(T) = 1$ and the minimality of T imply that $C_T(x) = 1$ for $x \in S^\#$. Hence, $|S| \mid h_y$ for all $y \in T^\#$.

(ii) The isomorphism theorems imply that $MC_G(x)/M \cong C_G(x)/C_G(x) \cap M = C_G(x)/C_M(x)$. Thus, $|M|/|C_M(x)| = |MC_G(x)|/|C_G(x)|$. Since $|G| = k|MC_G(x)|$ for some integer k, the result follows.

If γ is a nonempty set of primes and G is a solvable group, let G_γ denote a Hall γ-subgroup of G.

Lemma 2. *Assume $c \geq 1$, n is a positive integer, and $\mathscr{C}$ is a set of finite solvable groups closed under homomorphic images such that if $H \in \mathscr{C}$, then either $|\beta(H)| \leq c\alpha(H)$ or $|H| \geq n$. If $G \in \mathscr{C}$ with $|G| = n$ and $|\beta(G)| > c\alpha(G)$, then $|Z(G)| = 1$.*

Proof. Assume $|Z(G)| \neq 1$ and $P = Z(G)_p \neq 1$. Let $H \supseteq P$ where $H/P = Z(G/P)$, then $G/P \in \mathscr{C}$ and $|G/P| < n$ imply that $|\beta(G/P)| \leq c\alpha(G/P)$. If $p \in \beta(G/P)$ or $H_p = P$, it would follow that $\beta(G) = \beta(G/P)$ and $|\beta(G)| \leq c\alpha(G/P) \leq c\alpha(G)$. Hence, we may assume $G_p = H_p \neq P$.

$[G, G_p] = [G, H_p] \subseteq P$ and $[P, G] = 1$ imply that $[G_{p'}, G_p] = 1$ by [4, Theorem 5.3.2]. Hence, $G = G_{p'} \times G_p$ and $G_{p'} \cong G/G_p \in \mathscr{C}$. Let $x \in G_p$ with $h_x = p$, then $ph_y = h_{xy}$ for any $y \in G_{p'}$. Therefore $\alpha(G_{p'}) = \alpha(G) - 1$. Since $G_{p'} \in \mathscr{C}$, and $|G_{p'}| < n$, $|\beta(G_{p'})| \leq c\alpha(G_{p'}) = c(\alpha(G) - 1)$. However, $G = G_{p'} \times G_p$ yields $Z(G) = Z(G_{p'}) \times P$ whence $\beta(G) = \beta(G_{p'}) \cup \{p\}$ and $|\beta(G)| = |\beta(G_{p'})| + 1 \leq c\alpha(G) - c + 1 \leq c\alpha(G)$. This is a contradiction.

Proof of Theorem A. For any finite supersolvable group H, let H_1 denote the intersection of the maximal normal subgroups of H, and let $\mathscr{C} = \{H \mid H/H_1$ is cyclic$\}$. Clearly $\mathscr{C}$ is closed under homomorphic images so Lemma 2 applied with $c = 2$ implies we can assume $Z(G) = 1$ if G is a minimal counterexample to Theorem A.

Let A be a normal subgroup of G such that G/A is nilpotent, and suppose p is a prime dividing $[G : A]$. There is a normal subgroup K of G such that $G/K \simeq (G/A)_p$. If $B \supseteq K$ such that $B/K = \Phi(G/K)$, the Frattini subgroup of G/K, then $B \supseteq G_1$ whence G/B is cyclic. Therefore, G/A is cyclic and $A \geq G'$. In particular, it follows that $G = \langle x\rangle G'$ and if $\sigma = \{v \mid G$ does not have a normal v'-complement$\}$, then $\langle x_{\sigma'}\rangle$ is a Hall σ'-subgroup of G.

If $v \in \sigma$ and $v \nmid h_x$, then $G_v \subseteq C_G(x)$ for some Sylow v subgroup G_v of G. Hence, $\langle x\rangle \subseteq O_{v'}(G)V$ where V is a v-group such that $O_{v',v}(G) = O_{v'}(G)V$. $O_{v'}(G)V$ is a proper normal subgroup of G by the definition of σ. Hence, there is a maximal normal subgroup M such that $\langle x\rangle \subseteq O_{v'}(G)V \subseteq M$. However, $G' \subseteq M$ now yields $G \subseteq M$. Therefore, $v \mid h_x$ and $|\sigma| \leq \alpha(G)$. Since G is supersolvable, $G' \subseteq F(G)$ by Theorem 10.5.4 [5]. Let $x_1 = x_{\sigma'}$, then $\langle x_1\rangle = G_{\sigma'}$ and $G = \langle x\rangle G'$ imply that $\langle x_1\rangle (G')_\sigma \Delta G$. Further, $C_{\langle x_1\rangle}((G')_\sigma) = 1$ since $Z(G) = 1$. Hence, Lemma 1(i) applied to $\langle x_1\rangle (G')_\sigma$ implies $|\langle x_1\rangle| \mid h_y$ for some $y \in (G')_\sigma$. Therefore, $|\sigma'| \leq \alpha(G)$ and $\beta(G) = \sigma \cup \sigma'$ implies $|\beta(G)| \leq 2\alpha(G)$.

Proof of Theorem B. Let $\mathscr{C} = \{H \mid H$ is a finite supersolvable group such that $p \geq \alpha(H)$ for all prime divisors of $|H|\}$, then $\mathscr{C}$ is closed under homomorphic images. Assume G is a counterexample to Theorem B of minimum order, then applying Lemma 2 with $c = 2$, $Z(G) = 1$. Further, by Theorem A, there is a prime p and normal subgroup K of G such that G/K is of type (p, p) and there are $p+1$ distinct maximal normal subgroups M_i, $i = 1, \ldots, p+1$ such that $K = \bigcap_{i=1}^{p+1} M_i$ and $|G/M_i| = p$. It follows from Lemma 1(ii) that $M_i \in \mathscr{C}$ and $\alpha(M_i) \leq \alpha(G)$. Now $|M_i| < |G|$ implies $|\beta(M_i)| \leq 2\alpha(M_i)$. If $Z(M_i) = 1$, then $p \mid (|M_i|, |G/M_i|)$ whence $\beta(M_i) = \beta(G)$. Now $|\beta(G)| = |\beta(M_i)| \leq 2\alpha(M_i) \leq 2\alpha(G)$ yields a contradiction.

Therefore $Z(M_i)_{q_i} \neq 1$ for some prime q_i. If $q_i = p$, then $G = \langle x \rangle M_i$ with x a p-element implies $1 \neq C_{Z(M_i)_{q_i}}(x) \subseteq Z(G) = 1$. Hence, $q_i \neq p$ and $Z(M_i)_{q_i} = G_{q_i}$ is a normal Sylow q_i subgroup of G. $G = M_i M_j$ for $i \neq j$ yields $q_i \neq q_j$ if $i \neq j$. If $g \notin M_i$, then $G = \langle g \rangle M_i$ so that $Z(G) = 1$ yields $q_i \mid h_g$.

Let $G = \langle x \rangle M_1$ where x is a p-element and $x^p \in K$. Since G/K is of type (p, p), we can choose notation so that $x \notin \bigcup_{i=1}^{p} M_i$ and $x \in M_{p+1}$. Therefore, $\prod_{i=1}^{p} q_i \mid h_x$. However, $p \mid h_z$ for $z \in Z(M_{p+1})_{q_{p+1}}$. Therefore, $p \prod_{i=1}^{p} q_i \mid h_{xz}$, which contradicts $\alpha(G) < p+1$.

Proof of Theorem D. Let $\mathscr{C} = \{H \mid H$ is a finite solvable group and h_x is squarefree for all $x \in H\}$, then $\mathscr{C}$ is closed under homomorphic images. Let G be a minimal counterexample to Theorem D, then Lemma 2 with $c = 2$ yields $Z(G) = 1$.

We claim that if $H \in \mathscr{C}$, then H is supersolvable, i.e. H has cyclic chief factors. Proceeding by induction, it is sufficient to show $|R| = p$ if R is a minimal normal p-group. If $R \subseteq Z(H)$, the result is clear. If not, let $\overline{H} = H/C_H(R)$. By induction $|\overline{S}| = q$ if $\overline{S}$ is a minimal normal subgroup of $\overline{H}$. Hence $S = \langle x \rangle C_H(R)$ where $x^q \in C_H(R)$ for S the preimage of $\overline{S}$. Hence, $R \neq C_R(x) \Delta H$ and the minimality R imply that $C_R(x) = 1$. Now $R \Delta H$ implies $|R| \mid h_x$ whence h_x squarefree yields $|R| = p$.

Theorem 10.5.4 [5] now implies $H' \subseteq F(H)$ if $H \in \mathscr{C}$. If $H/F(H)$ is not cyclic, there is a prime v and a normal subgroup $K \supseteq F(H) \supseteq H'$ such that $H = VK$, and $V/V \cap K \simeq H/K \simeq (H/F(H))_v$ is a noncyclic v group and V normalizes some $H_{v'} \subseteq K$. Now H' nilpotent implies $R = F(H)_{v'}$ is nilpotent and $H' \subseteq RC_H(R)$. Let $\overline{H} = H/RC_H(R)$, then $\overline{H}$ is Abelian. Hence, $\overline{V} \Delta \overline{H}$ and Lemma 1 applied to the semidirect product of $\overline{V}$ and R now implies that there is a $y \in R$ such that $|\overline{V}| \mid h_y$. Now h_y squarefree implies $|V/V_1| = v$ where $V_1 = C_V(R)$. Since $H/F(H)$ is abelian it now follows that $V_1 \subseteq C_G(H_{v'}) \subseteq O_v(H) = F(H)_v$. This contradicts $(H/F(H))_v$ noncyclic. Thus, $H = \langle z \rangle F(H)$.

Let $G \in \mathscr{C}$ be of minimal order such that $|\beta(G)| > 2\alpha(G)$ and let $p \in \pi(|F(G)|)$. Now $Z(G) = 1$ and $G = \langle z \rangle F(G)$ imply that $p \mid h_z$. Hence, $|\pi(|F(G)|)| \leq \alpha(G)$. Let $\sigma = \{q \mid q \mid |\langle z \rangle|$ and $q \notin \pi(|F(G)|)\}$ then $\langle z_1 \rangle = \langle z \rangle_\sigma$ is an abelian $\pi(|F(G)|)'$ group acting faithfully on $F(G)$ since $Z(G) = 1$. Hence, by Lemma 1 there is a $y \in F(G)$ with $|\langle z_1 \rangle| \mid h_y$. Therefore $|\sigma| \leq \alpha(G)$ and $|\beta(G)| = |\pi(|F(G)|) \cup \sigma| \leq 2\alpha(G)$.

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