PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 113, Number 2, October 1991

CREATION AND ANNIHILATION OPERATORS FOR ANHARMONIC OSCILLATORS

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(Communicated by Palle E. T. Jorgensen)

ABSTRACT. Positive elliptic operators from the Helffer-Robert classes of pseudodifferential operators on the line can be approximately factored as products of creation and annihilation operators. Operators of anharmonic oscillators belong to these classes.

1. INTRODUCTION

The spectral problem for the harmonic oscillator

$$H_g = -\frac{d^2}{dx^2} + x^2$$

can be treated in a purely algebraic way. The key observation is that the operator H_g admits the representation

$$H_g = a^+a^- + 1, \tag{1.1}$$

where $a^{\pm}$, the *creation* and *annihilation* operators, satisfy the commutation relation

$$[a^-, a^+] = 2. \tag{1.2}$$

Analytically, the operators $a^{\pm}$ can be represented as

$$a^{\pm} = \mp\frac{d}{dx} + x.$$

Note that

$$a^+ = a^{-*}. \tag{1.3}$$

Let $\lambda_1 < \lambda_2 < \cdots$ be the set of eigenvalues of the operator H_g, and $\{\phi_1, \phi_2, \dots\}$ be the set of normalized eigenfunctions that correspond to these eigenvalues. Then

$$a^-\phi_{j+1} = h_j\phi_j, \qquad a^+\phi_j = h_j\phi_{j+1}, \quad \text{and} \quad a^-\phi_1 = 0. \tag{1.4}$$

Received by the editors April 18, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47G05; Secondary 47A10, 47E05, 58F19.

The relations (1.1) and (1.2) imply that

$$\lambda_{j+1} = \lambda_j + 2. \tag{1.5}$$

The relations (1.4) and (1.5) allow us to compute all eigenvalues and all eigenfunctions. Note that

$$\operatorname{ind} a^- = 1. \tag{1.6}$$

The aim of this paper is to implement the same idea to more general operators, for example, to the anharmonic oscillator

$$H_a = -\frac{d^2}{dx^2} + x^2 + gx^4, \qquad g > 0.$$

First, how to compute the order of the operator H_a? The following discussion is a part of mathematical folklore. The "right" order does not equal, of course, the order of the highest derivative (2) — the operator H_a is unitary equivalent to

$$g\frac{d^4}{dx^4} - \frac{d^2}{dx^2} + x^2$$

via the Fourier transform, and the "right" order must be invariant under unitary equivalence. To compute the order one can make three assumptions:

(i) the order of the commutator of two operators equals the sum of their orders minus one;
(ii) operators $-d^2/dx^2$ and x^4 are of the same order;
(iii) the operator of multiplication by a nonzero constant has the order 0.

These assumptions imply $\operatorname{ord} x = 1/3$, $\operatorname{ord}(d/dx) = 2/3$, and $\operatorname{ord} H_a = 4/3$. The order of H_a, having been computed in such a way, agrees with the "spectral order":

$$\lambda_j(H_a) \sim \operatorname{const} j^{4/3}.$$

We can now define the creation and the annihilation operators for the anharmonic oscillator by the formulas (1.4), where ϕ_j are the eigenfunctions of H_a, and we have a freedom in choosing the constants $h_j \neq 0$. The relations (1.1) and (1.2) with H_a instead of H_g are not true any more, but operators H_a, a^+a^-, and $[a^+, a^-]$ are mutually commuting.

These observations do not give us any information so long as we are not able to visualize the operators $a^{\pm}$.

I shall work within the classes of pseudodifferential operators introduced by Helffer and Robert [1]. I change the notations and the normalization of orders. Let $0 < \alpha < 1$ be a real number, and let m be a complex number. The class S^m_α of symbols consists of all C^∞-functions $p(x, \xi)$ that satisfy the following estimates:

$$|\partial_x^a \partial_\xi^b p(x, \xi)| \leq C(a, b)(1 + |x|^{1/\alpha} + |\xi|^{1/(1-\alpha)})^{\Re m - a\alpha - b(1-\alpha)}.$$

For example,

$$\xi^2 + x^4 \in S^{4/3}_{1/3}.$$

We define α-homogeneous symbols as $C^\infty(R^2 - (0, 0))$-functions that satisfy the following homogeneity property:

$$p(\tau^\alpha x, \tau^{1-\alpha}\xi) = \tau^l p(x, \xi).$$

The number l is called the order of α-homogeneity ($\mathrm{ord}_\alpha p$). Let q be a positive integer. The class $\mathrm{CS}^m_{\alpha,q}$ of classical symbols consists of all symbols from S^m_α that admit asymptotic expansion

$$p(x, \xi) \sim \sum_{j=0}^{\infty} p_j(x, \xi),$$

where

$$\mathrm{ord}_\alpha p_j = m - \frac{j}{q}.$$

The class of pseudodifferential operators

$$Pu(x) = \frac{1}{2\pi}\int e^{i(x-y)\xi} p(x, \xi)u(y)\, dy\, d\xi$$

with $p(x, \xi) \in \mathrm{CS}^m_{\alpha,q}$ will be denoted by $\mathrm{CL}^m_{\alpha,q}$. Helffer and Robert proved in [1] all usual theorems about pseudodifferential operators for the classes $\mathrm{CL}^m_{\alpha,q}$. An operator $P \in \mathrm{CL}^m_{\alpha,q}$ is elliptic if its principal symbol $p_0(x, \xi)$ is nonvanishing for all $(x, \xi) \neq (0, 0)$. The theory of complex powers for an elliptic operator from $\mathrm{CL}^m_{\alpha,q}$, $m > 0$, was built in [1]. The theory of complex powers has been built under the usual assumption that an elliptic operator has a principal ray. It means that an angle $|\arg\lambda - \theta| < \varepsilon$ is free from values of the principal symbol of the elliptic operator for some θ and ε. To be definite, I shall assume $\theta = \pi$.

The last fact I would like to mention about operators from the class $\mathrm{CL}^m_{\alpha,q}$ is the characterization of operators of order $-\infty$ (negligible operators). Negligible operators are operators with smooth Schwartz's kernels satisfying the following estimates

$$|\partial_x^a \partial_y^b K(x, y)| \leq C(a, b, N)(1 + |x| + |y|)^{-N}$$

for every a, b and N.

In this paper I shall prove the following:

Theorem. *Let $H \in \mathrm{CL}^m_{\alpha,q}$ is an elliptic pseudodifferential operator with positive principal symbol, and $m > 0$. Then there exist operators $a^\pm \in \mathrm{CL}^{m/2}_{\alpha,q}$ (creation and annihilation operators), such that*

(i) $\mathrm{ord}(a^+ - a^{-*}) < m$;
(ii) $H = a^+ a^- + R_1$;
(iii) $[a^-, a^+] = f(H) + R_2$; *and*
(iv) $\mathrm{ind}\, a^- = 1$,

where R_1 *and* R_2 *are negligible operators, and*

$$(1.7)\qquad f(\lambda) \sim \sum_{j=0}^{\infty} c_j \lambda^{\sigma_j}, \qquad \lambda \to \infty,$$

with

$$\sigma_j = \frac{m - 1 - j/q}{m}.$$

The symbols of the operators $a^{\pm}$ and the constants c_j are local expressions of the symbol of the operator H. If the operator H is selfadjoint, and if the spectrum of this operator is asymptotically simple (meaning that the operator H has at most a finite number of multiple eigenvalues), then one can choose $a^{\pm}$ in such a way that $R_2 = 0$. In this case

$$(1.8)\qquad \lambda_{j+1} - \lambda_j \sim \sum_{j=0}^{\infty} c_k \lambda_j^{\sigma_k}, \qquad j \to \infty.$$

Helffer and Robert derive in [2] the complete asymptotic expansion for the eigenvalues of selfadjoint elliptic pseudodifferential operators from the class $\mathrm{CL}^m_{\alpha,q}$. Their expansion includes a nonlocal term, and for the differences of eigenvalues it allows one to write down only the first term explicitly.

2. Proof of the theorem

Let

$$h(x, \xi) \sim \sum_{j=0}^{\infty} h_j(x, \xi)$$

be the asymptotic expansion of the symbol of the operator H;

$$\mathrm{ord}_\alpha h_j = m - \frac{j}{q}.$$

The principal symbol h_0 is positive. Let

$$a^{\pm}(x, \xi) \sim \sum_{j=1}^{\infty} a_j^{\pm}(x, \xi)$$

be asymptotic expansions of the symbols of the operators $a^{\pm}$,

$$\mathrm{ord}_\alpha a_j^{\pm} = \frac{m}{2} - \frac{j}{q}.$$

I shall find the functions $a_j^{\pm}$ and the constants c_j from (1.7) step by step.

At the first step we shall construct the principal symbols $a_0^{\pm}$. The property (i) of the operators $a^{\pm}$ (see the theorem) implies

$$a_0^+ = \overline{a_0^-}.$$

The principal symbol of the operator a^+a^- equals $|a_0^-|^2$. Therefore,

$$|a_0^-(x, \xi)| = (h_0(x, \xi))^{1/2},$$

and

$$a_0^-(x, \xi) = (h_0(x, \xi))^{1/2} e^{i\phi(x, \xi)}, \tag{2.1}$$

where $\phi(x, \xi)$ is a real α-homogeneous function of order 0. The principal symbol of the commutator $[a^-, a^+]$ equals

$$\frac{1}{i}\left\{a_0^-(x, \xi), \overline{a_0^-(x, \xi)}\right\} = \{\phi, h_0\}, \tag{2.2}$$

where

$$\{f(x, \xi), g(x, \xi)\} = \frac{\partial f}{\partial \xi}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial \xi}$$

is the Poisson bracket. The principal symbol of the operator $f(H)$ equals $c_0[h_0(x, \xi)]^{\sigma_0}$, so the constant σ_0 (see (1.7)) is forced to be equal to $(m-1)/m$. Now we can write

$$\{\phi, h_0\} = c_0 h_0^{(m-1)/m}. \tag{2.3}$$

The operator $\{\cdot, h_0\}$ is an operator of differentiation along the vector field, which is tangent to the lines $h_0 = \text{const}$. One can find the values of the function ϕ on the curve $C = \{h_0(x, \xi) = 1\}$, and then one can extend this function by α-homogeneity. On the curve C

$$\phi = c_0 \tau, \tag{2.4}$$

where τ is a normal parameter along C:

$$\begin{aligned} \frac{dx}{d\tau} &= -\frac{dh_0}{d\xi}; \\ \frac{d\xi}{d\tau} &= \frac{dh_0}{dx}. \end{aligned} \tag{2.5}$$

To fix the normal parameter we have to impose initial conditions for (2.5). To be definite, let us take

$$x(0) > 0, \qquad \xi(0) = 0.$$

the value of the normal parameter is defined up to the period T, and a simple computation shows that

$$T = \frac{1}{m}\int_{h_0(x, \xi) \le 1} dx\, d\xi. \tag{2.6}$$

The function $\exp(i\phi)$ is defined if

$$c_0 T = 2\pi k,$$

where k is an integer. Now we have to specify the value of k.

I claim that k equals exactly the index of the operator a^-. If $k = 0$, one can deform the principal symbol of the operator a^- to $[h_0]^{1/2} > 0$ in the class of elliptic symbols from $\mathrm{CL}_{\alpha, q}^{m/2}$. Therefore, in this case $\operatorname{ind} a^- = 0$. Let $k \neq 0$. First, let us multiply the operator a^- by $H^{|k|/m - 1/2}$. This operation does not change the index, and the resulting operator b is of order $|k|$. Then let us deform the curve C to the unit circle S. The principal symbol of the operator b_1 on S is defined as the image of the principal symbol of b on C, and it is extended to all points (x, ξ) by α-homogeneity of order $|k|$. Then one can deform the phase of the principal symbol of b_1 to $k\theta$, where θ is the polar angle. Denote the resulting operator by c_0. Now we introduce the deformation c_t, $0 \le t \le 1$. Operators c_t belong to the classes $\mathrm{CL}^{|k|}_{(1-t)\alpha+t/2,q}$; the principal symbols of these operators equal the principal symbol of c_0 on the unit circle S, and they are extended to all points (x, ξ) by $\alpha(t) = (1-t)\alpha + t/2$-homogeneity of order $|k|$.

Now I want to show that $\operatorname{ind} c_t = \text{const}$. The principal symbol of the operator c_t equals $u_t(x, \xi)\exp(i\phi_t(x, \xi))$, where the function u_t is positive, and the function ϕ_t is $\alpha(t)$- homogeneous. Let d_t be an operator with the principal symbol $\exp(i\phi_t)$. Clearly, $\operatorname{ind} c_t = \operatorname{ind} d_t$. Let $\rho = \min\{\alpha, 1-\alpha\}$. Then all functions $\exp(i\phi_t)$ belong to the space Γ^0_ρ of symbols [3]. It means that

$$|\partial_x^m \partial_\xi^n e^{i\phi(x,\xi)}| \le C_{m,n}(1 + |x| + |\xi|)^{-(m+n)\rho}.$$

These symbols are elliptic, and they depend on t continuously. Therefore, $\operatorname{ind} d_t = \text{const}$, and $\operatorname{ind} c_t = \text{const}$.

The principal symbol of the operator c_1 equals $(x + i\xi)^k$. This expression equals exactly the symbol of the kth power of the annihilation operator for the harmonic oscillator if k is positive, and it equals the symbol of the $-k$th power of the creation operator for the harmonic oscillator if k is negative. Therefore,

$$\operatorname{ind} c_1 = k,$$

and

$$\operatorname{ind} a^- = k.$$

We want to construct the annihilation operator with the index 1; hence $k = 1$, and

$$c_0 = \frac{2\pi}{T}.$$

At this point we have constructed the principal symbol of the operator a^-. This principal symbol is defined by the conditions (i)–(iv) uniquely up to the phase shift

$$\phi \mapsto \phi + \text{const}.$$

This phase shift corresponds to taking different initial conditions for (2.5).

The functions $a_j^\pm$ and the constants c_j can be constructed inductively. Assume that all the functions $a_k^\pm$ and the constants c_k, $k < j$, have been defined. Then the jth term in the symbol of the operator a^+a^- equals $a_j^+ a_0^- + \overline{a_0^-} a_j^-$ plus the expression, which depends on $a_k^\pm$, $k < j$ only. This symbol must be equal to h_j. Therefore,

$$a_j^+ a_0^- + \overline{a_0^-} a_j^- = g_j,$$

where g_j is a known expression. Let

$$a_j^- = \alpha_j e^{i\phi} \quad \text{and} \quad a_j^+ = \beta_j e^{-i\phi}.$$

Then

$$h_0^{1/2}(\alpha_j + \beta_j) = g_j,$$

and

$$\alpha_j + \beta_j = u_j. \tag{2.7}$$

The jth component of the symbol of the operator $[a^-, a^+]$ equals

$$\frac{1}{i}(\{a_0^-, a_j^+\} + \{a_j^-, a_0^+\}) \tag{2.8}$$

plus the sum of the terms, which depend on $a_k^\pm$, $k < j$, only. The expression (2.8) equals

$$\begin{aligned} &\frac{1}{i}\{h_0^{1/2}e^{i\phi}, \beta_j e^{-i\phi}\} + \frac{1}{i}\{\alpha_j e^{i\phi}, h_0^{1/2}e^{-i\phi}\} \\ &\quad = \frac{1}{i}\{h_0^{1/2}, \beta_j - \alpha_j\} + h_0^{1/2}\{\phi, \alpha_j + \beta_j\} + (\alpha_j + \beta_j)\{\phi, h_0^{1/2}\}. \end{aligned} \tag{2.9}$$

Only the first term on the right-hand side of (2.9) is unknown. We know the coefficients c_k, $k < j$, in the right-hand side of (1.7), and the jth term in the asymptotic expansion of the symbol of the operator $f(H)$ equals $c_j h_0^{\sigma_j}$ plus a sum of known terms. Therefore,

$$\frac{1}{i}\{h_0^{1/2}, \alpha_j - \beta_j\} = -c_j h_0^{\sigma_j} + v_j, \tag{2.10}$$

where the function v_j is known. One can solve equation (2.10) on the curve $C = \{h_0(x, \xi) = 1\}$, and then extend the solution by homogeneity. On the curve C, the equation (2.10) has the form

$$\frac{1}{2i}\{h_0, \alpha_j - \beta_j\} = -c_j + v_j. \tag{2.11}$$

The equation (2.11) has a solution iff

$$\int_C (-c_j + v_j(x, \xi))\, d\tau = 0,$$

where τ is the normal parameter along C (see (2.5)). One can treat the form $d\tau$ as the inner product of the Hamilton vector field, corresponding to the hamiltonian h_0, with the standard symplectic form $dx \wedge d\xi$. Thus,

$$c_j = \frac{1}{L}\int_C v_j(x, \xi)\, d\tau. \tag{2.12}$$

The equality (2.11) defines the value of c_j. Now (2.10) can be solved, and its solution is defined up to a constant. Combining a solution of this equation with (2.7), we can determine both α_j and β_j.

The operators $a^{\pm}$ with symbols $a^{\pm}(x, \xi)$ satisfy the properties (i)–(iv) of the theorem.

Let the operator H be selfadjoint. I shall eliminate the additional term R_2 in (iii). Let $\{\lambda_j\}$ be the set of all eigenvalues of the operators H, $\lambda_1 \leq \lambda_2 \leq \cdots$, and $\{\psi_j\}$ be an orthonormal set of eigenfunctions that correspond to λ_j. By R_j I shall denote negligible operators. First,

$$\begin{aligned} Ha^{+}\psi_j &= (a^{+}a_{-} + R_1)a^{+}\psi_j = a^{+}H\psi_j + a^{+}[a^{-}, a^{+}]\psi_j + R_3\psi_j \\ &= a^{+}(H + f(H))\psi_j + R_4\psi_j = (\lambda_j + f(\lambda_j))\psi_j + R_4\psi_j. \end{aligned} \tag{2.13}$$

The operator R_4H^{-N} is bounded for every N. Therefore,

$$\|R_4\psi_j\| \leq C_N\lambda_j^{-N},$$

and

$$\|[H - (\lambda_j + f(\lambda_j))]a^{+}\psi_j\| \leq C_N\lambda_j^{-N}.$$

On the other hand,

$$\|a^{+}\psi_j\| \sim \lambda_j^{1/2},$$

because the operator $(a^{+})^{*}a^{+}$ differs from H by an operator of an order smaller than $m = \operatorname{ord} H$. Therefore,

$$\|[H - (\lambda_j + f(\lambda_j))]a^{+}\psi_j\| \leq C_N\lambda_j^{-N} \tag{2.14}$$

for arbitrary N. The inequality (2.14) implies that the closest to $a^{+}\psi_j$ eigenfunction χ_j of the operator H is at the distance $O(\lambda_j^{-\infty})$ from $a^{+}\psi_j$, and the corresponding eigenvalue is at the distance $O(\lambda_j^{-\infty})$ from $\lambda_j + f(\lambda_j)$. We define the operator a_1^{+} by

$$a_1^{+}\psi_j = \chi_j.$$

Clearly, the difference $a^{+} - a_1^{+}$ is a negligible operator. In the same way one can redefine the annihilation operator by adding a negligible operator in such a way that the new operator a_1^{-} maps functions ψ_j into eigenfunctions of H. Obviously, all statements (i)–(iv) are satisfied for $a_1^{\pm}$. I shall denote these operators $a_1^{\pm}$ by $a^{\pm}$.

For large values of λ the function $\lambda + f(\lambda)$ is increasing, and, therefore,

$$a^{+}\psi_j = h_j\psi_{k(j)} \quad \text{and} \quad a^{-}\psi_j = h_j'\psi_{l(j)},$$

where $k(j)$ is increasing, and $l(j)$ is decreasing. Now I am going to prove that

$$k(j) = j + 1 \quad \text{and} \quad l(j) = j - 1$$

for large values of j.

First, for large j both functions $k(j)$ and $l(j)$ are one-to-one. In fact, if $l(j) = l(j')$, then either $k(l(j)) \neq j$ or $k(l(j')) \neq j'$. Let $k(l(j)) \neq j$. In this case

$$\|a^{+}a^{-}\psi_j - H\psi_j\| \geq \|H\psi_j\|,$$

because the system $\{\psi_j\}$ is orthogonal. On the other hand,

$$\frac{\|(a^+a^- - H)\psi_j\|}{\|H\psi_j\|} \to 0, \qquad j \to \infty,$$

since the difference $a^+ - a^-$ is subordinated to a power H^a, $a < 1$ (in fact, this difference is subordinated to H^a for arbitrary a). The same arguments can be applied to the operator a^+. In this case one has to take the operator a^-a^+ instead of a^+a^-, and the difference $a^-a^+ - H$ is subordinated to $H^{(m-1)/m}$. We have proved that $k(l(j)) = j$ and $l(k(j)) = j$ for $j \geq j_0$.

Assume that $k(j) > j+1$ for some $j > j_0$. Let us restrict the operator a^+ to the invariant subspace of $L^2(R)$ spanned by the functions ψ_m, $m \geq j$. I shall denote this restriction by a_j^+. Clearly,

$$\operatorname{ind} a^+ = \operatorname{ind} a_j^+ \geq k(j) - j > 1.$$

On the other hand, $\operatorname{ind} a^+ = 1$. Therefore, $k(j) = j + 1$. The assertion about $l(j)$ follows from the fact that k and l are inverse functions for large values of j.

For the small values of j, $j < j_0$, one can redefine the operators $a^\pm$, say, by the formulas

$$a^+\psi_j = \psi_{j+1}, \qquad a^-\psi_j = \psi_{j-1}.$$

Such a change can result only in the addition of negligible operators to $a^\pm$. Now the negligible operator R_2 in (iii) equals 0. The formula (1.8) follows from (1.7). □

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