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A COMMUTATIVE BANACH ALGEBRA WITH FACTORIZATION OF ELEMENTS BUT NOT OF PAIRS

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ABSTRACT. We find a one-point Gleason part ϕ off the Šilov boundary of $H^{\infty}(\Delta)$ such that the maximal ideal M_{ϕ} factors but such that pairs do not factor in M_{ϕ}.

1. INTRODUCTION

A topological algebra A *factors* if, for each $y \in A$, there exist $a, x \in A$ with $y = ax$. We say that pairs factor in A if, for each $y_1, y_2 \in A$ there exist $a \in A$ and $x_1, x_2 \in A$ with $y_1 = ax_1$ and $y_2 = ax_2$. Also, A strongly factors if, for each sequence (y_n) tending to zero, there exist $a \in A$ and a sequence (x_n) tending to zero with $y_n = ax_n$. A *bounded approximate identity* for a Banach algebra A is a net $(e_\alpha) \subset A$ such that $xe_\alpha \to x$ and $e_\alpha x \to x$ for each $x \in A$ and such that $\sup \|e_\alpha\| < \infty$. It is well known that if A has a bounded approximate identity, then it strongly factors and hence factors [1, p. 62]. But a commutative Banach algebra may factor even if it does not have a bounded approximate identity: a nonseparable example is given in [4, p. 166] and it is easy to see that this example strongly factors. More recently, Willis gave an example of a commutative, separable Banach algebra which strongly factors, but has no bounded approximate identity, [10].

In this paper, we give an example of a commutative, nonseparable Banach algebra which factors, but does not strongly factor. More precisely, our algebra contains two elements with no common factor.

2. PRELIMINARIES

Throughout, X will denote a compact Hausdorff space and $C(X)$ the algebra of all continuous functions on X with the uniform norm

$$\|f\|_X = \sup\{|f(x)| : x \in X\}.$$

Also, if $S \subset X$, we let $\|f\|_S = \sup\{|f(x)| : x \in S\}$.

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Definition 2.1. A *uniform algebra* on X is a closed subalgebra of $C(X)$ that contains the constants and separates points of X.

Let A be a uniform algebra. For $x \in X$, we shall denote by M_x the maximal ideal $M_x = \{f \in A : f(x) = 0\}$ that consists of functions in A vanishing at x.

Definition 2.2. A *boundary* for a uniform algebra A is a subset S of X on which the modulus of every function in A assumes its maximum.

Among all closed boundaries for A there is a smallest one [9, 7.4] which is called the *Šilov boundary.*

Definition 2.3. A point $x \in X$ is called a *peak point* for A if there exists $f \in A$ such that $f(x) = \|f\|_X = 1$ and $|f(y)| < 1$ for $y \neq x$. It is called a *strong boundary point* for A if for each neighborhood U of x there exists $f \in A$ such that $f(x) = \|f\|_X = 1$ and $|f(y)| < 1$ for $y \in X \backslash U$.

It is easily seen that a peak point is a strong boundary point and a strong boundary point necessarily belongs to the Šilov boundary. Also, if X is metrizable, then $C(X)$ is separable and so is every closed subalgebra of $C(X)$. In this case, peak points and strong boundary points coincide. The following theorem relates maximal ideals and strong boundary points for uniform algebras.

Theorem 2.4. *The maximal ideal M_x of a uniform algebra A has a bounded approximate identity if and only if x is a strong boundary point.*

Proof. See [2, p. 101] and [3, 5.8].

3. Parts and maximal ideals in $H^\infty(\Delta)$

Let Δ be the open unit disc in the complex plane and let $H^\infty(\Delta)$ be the algebra of bounded analytic functions on Δ. Under the norm

$$\|f\| = \sup\{|f(z)| : |z| < 1\},$$

$H^\infty(\Delta)$ is a commutative Banach algebra with identity. Let Φ be the maximal ideal space of $H^\infty(\Delta)$ and consider the Gelfand map $f \mapsto \hat{f}$ from $H^\infty(\Delta)$ to $C(\Phi)$ defined by $\hat{f}(\phi) = \phi(f)$ for $\phi \in \Phi$. It is not hard to see that this is an isometry and therefore $H^\infty(\Delta)$ is isometrically isomorphic to its image $\widehat{H}^\infty(\Delta)$, a uniformly closed subalgebra of $C(\Phi)$ that separates points of ϕ and contains the constants. Hence $H^\infty(\Delta)$ can be regarded as a uniform algebra on the compact space Φ.

For ϕ and ψ in Φ, we consider the pseudohyperbolic metric

$$\rho(\phi, \psi) = \sup\{|\hat{f}(\phi)| : \|f\| \leq 1 \text{ and } \hat{f}(\psi) = 0\}, \tag{1}$$

and the relation $\phi \sim \psi$ if and only if $\rho(\phi, \psi) < 1$. This defines an equivalence relation [5, p. 142].

Definition 3.1. The equivalence classes of Φ under the equivalence relation $\sim$ are called *Gleason parts* or simply *parts* for $H^\infty(\Delta)$.

Definition 3.2. An *analytic disc* is the image of a one-to-one continuous map L from Δ to Φ such that $\hat{f} \circ L$ is analytic for each $f \in H^\infty(\Delta)$.

It is easily seen that the evaluation maps on Δ form an analytic disc.

Gleason parts for the algebra $H^\infty(\Delta)$ have been studied by Hoffman [8] where the next theorem is proved. See also [9, 17.1].

Theorem 3.3. *Each part for the algebra $H^\infty(\Delta)$ is either a one-point part or an analytic disc.*

Theorem 3.4. *If f is a bounded analytic function on Δ and if $\alpha_1, \alpha_2, \alpha_3, \ldots$ are the zeros of f, each repeated as often as its multiplicity, then*

$$\sum_{n=1}^{\infty} (1 - |\alpha_n|) < \infty. \tag{2}$$

Also, if (α_n) is a sequence in Δ satisfying (2) and such that $\alpha_n \neq 0$ for all n, if k is a nonnegative integer, and if

$$B(z) = z^k \prod_{n=1}^{\infty} \left(\frac{|\alpha_n|}{\alpha_n} \frac{\alpha_n - z}{1 - \overline{\alpha}_n z} \right) \qquad (|z| < 1) \tag{3}$$

then $B \in H^\infty(\Delta)$ and B has no zeros except at the points α_n, and at the origin if $k > 0$. See [7, p. 63].

The function B of the theorem is called a *Blaschke product.*

Definition 3.5. An *inner function* $f(z)$ is a bounded analytic function such that $|f(e^{i\theta})| = 1$ almost everywhere on the unit circle.

Theorem 3.6 [7, p. 66]. *Let f be a nonzero function in $H^\infty(\Delta)$. Then f is uniquely expressible in the form $f = Bg$, where B is a Blaschke product and g a bounded analytic function without zeros.*

Theorem 3.7 [6, p. 194]. *Let ϕ be a complex homomorphism on $H^\infty(\Delta)$. Then the following are equivalent:*

(i) *ϕ lies in the Šilov boundary for $H^\infty(\Delta)$;*
(ii) *$|\phi(B)| > 0$ for every Blaschke product B;*
(iii) *$|\phi(u)| = 1$ for every inner function u.*

For lack of a suitable reference, we include a proof of the following well-known result (see [5, p. 162, Exercise 3], and [6, p. 438]). We will need the function f of the proof for our example.

Theorem 3.8. *There exists a one-point part ϕ off the Šilov boundary of $H^\infty(\Delta)$ and a function $f \in H^\infty(\Delta)$ such that f has no zeros in Δ and such that $f \in M_\phi$.*

Proof. We consider the function defined by

$$f(z) = \exp\left(\frac{z+1}{z-1}\right) \qquad (|z| < 1). \tag{4}$$

Clearly $|f(z)| < 1$ for z in Δ. Also, if we look at the boundary behavior of f we see that $|f(z)| = 1$ for all $z \neq 1$ on the unit circle and this leads us to say that $\|f\| = 1$. Let $Z(\hat{f}) = \{\phi \in \Phi: \hat{f}(\phi) = 0\}$. Since $f(z)$ tends to zero as z tends to 1 along the real axis, $\hat{f}$ has no inverse and so $Z(\hat{f})$ is nonempty. Furthermore, since $f(z)$ is an inner function and $\phi(f) = 0$, then by Theorem 3.7 we have $Z(\hat{f}) \cap \Gamma = \varnothing$, where Γ is the Šilov boundary of $H^\infty(\Delta)$.

We next show that $Z(\hat{f})$ is a union of parts. Let $\phi \in Z(\hat{f})$ and let θ be in the same Gleason part as ϕ with $\theta(f) \neq 0$. Then, if f_n is the nth root of f, which clearly belongs to $H^\infty(\Delta)$, we have

$$\rho(\phi, \theta) \geq \sup_n |f_n(\theta)| = \sup_n |\hat{f}(\theta)|^{1/n} = \lim_n |\hat{f}(\theta)|^{1/n} = 1. \tag{5}$$

This is a contradiction because ϕ and θ lie in the same part. This shows that $\hat{f} = 0$ on the full Gleason part containing ϕ. So for each $\phi \in Z(\hat{f})$, the Gleason part containing ϕ is contained in $Z(\hat{f})$. Thus $Z(\hat{f})$ is a union of parts.

To complete the proof of the theorem, we suppose that $Z(\hat{f})$ does not contain a one-point part, and so by Theorem 3.3 is a union of analytic discs, and we seek a contradiction.

Let $\mathscr{F}$ be the family of nonempty, closed subsets of $Z(\hat{f})$ which are a union of analytic discs. This family is nonempty because $Z(\hat{f})$ is already one of its elements. Let $\{F_\lambda : \lambda \in \Lambda\}$ be a decreasing chain in $\mathscr{F}$ and consider $G = \bigcap_{\lambda \in \Lambda} F_\lambda$. Clearly G is nonempty because it is the intersection of closed subsets which have the finite intersection property in a compact space. Also, if $\phi \in G$, then $\phi \in F_\lambda$ and the part containing ϕ is included in F_λ for each λ because F_λ is a union of parts. This shows that G is an element of the family $\mathscr{F}$, and so Zorn's lemma asserts the existence of a minimal element F_0 in $\mathscr{F}$.

Since F_0 is not a singleton, and since $H^\infty(\Delta)$ separates points of Φ, there exists a function g in $H^\infty(\Delta)$ such that g is not constant on F_0. We can suppose that $|g|$ is not constant on F_0. Let

$$F_1 = \left\{ \phi \in F_0 : |\hat{g}(\phi)| = \sup_{\psi \in F_0} |\hat{g}(\psi)| \right\}. \tag{6}$$

Then F_1 is a nonempty, proper closed subset of F_0. Let ϕ be in F_1 and let Π be the analytic disc to which ϕ belongs. Let L be a one-to-one map from the open unit disc onto the Gleason part Π with $\hat{g} \circ L$ analytic for all $g \in H^\infty(\Delta)$. There exists z_0 in Δ such that $L(z_0) = \phi$. Then

$$|\hat{g} \circ L(z_0)| = |\hat{g}(\phi)| = \sup_{\psi \in F_0} |\hat{g}(\psi)| \geq \sup_{\psi \in \Pi} |\hat{g}(\psi)| = \sup_{z \in \Delta} |\hat{g} \circ L(z)|. \tag{7}$$

This shows that $|\hat{g} \circ L(z_0)| = \sup_{z \in \Delta} |\hat{g} \circ L(z)|$ and by the maximum modulus principle we have that $|\hat{g} \circ L|$ is constant on Δ and $|\hat{g}|$ is constant on Π with value $|\hat{g}(\phi)|$. Thus, $\Pi \subset F_1$ and since the choice of ϕ in F_1 was arbitrary, F_1 is a union of parts and $F_1 \in \mathscr{F}$. This contradicts the minimality of F_0 and the proof of the theorem is complete.

4. Existence of a pair in M_ϕ with no common factor

Theorem 4.1 [6, p. 412]. *Let ϕ be an element of Φ. The following are equivalent:*

(i) *the Gleason part containing ϕ is a one-point part;*
(ii) *if $f \in M_\phi$ with $\|f\| \le 1$, then there exist $f_1, f_2 \in M_\phi$ with $f = f_1 f_2$ and $\|f_j\| \le 1$.*

Proposition 4.2. *Let ϕ be a one-point part off the Šilov boundary of $H^\infty(\Delta)$. Then the maximal ideal $M_\phi = \{f \in H^\infty(\Delta): \phi(f) = 0\}$ factors, but has no bounded approximate identity.*

Proof. Theorem 4.1 shows that M_ϕ factors. Also, since ϕ does not belong to the Šilov boundary, ϕ cannot be a strong boundary point. By Theorem 2.4, M_ϕ has no bounded approximate identity.

We are now ready to show that in the maximal ideal M_ϕ pairs do not factor and hence M_ϕ does not strongly factor.

Theorem 4.3. *There exists a one-point Gleason part ϕ off the Šilov boundary of $H^\infty(\Delta)$, such that the maximal ideal M_ϕ factors but pairs do not factor in M_ϕ.*

Proof. Let ϕ and f satisfy the conditions of Theorem 3.8. Theorem 3.7 tells us that there exists a Blaschke product B_0 in the maximal ideal M_ϕ. To obtain a contradiction, suppose that B_0 and f have a common factor, that is, there exist g, h_1, and h_2 in M_ϕ with

$$B_0 = gh_1 \quad \text{and} \quad f = gh_2. \tag{8}$$

Since f has no zeros in the open disc, it follows that the zero set of g, $Z(g) = \{z \in \Delta: g(z) = 0\}$, is empty. We now factor h_1 as in Theorem 3.6, say $h_1 = \widetilde{B}\tilde{h}_1$, where $\widetilde{B}$ is a Blaschke product and $\tilde{h}_1$ is a bounded analytic function without zeros. We have

$$B_0 = g\widetilde{B}\tilde{h}_1 = \widetilde{B}(g\tilde{h}_1). \tag{9}$$

Since $Z(g\tilde{h}_1) = Z(g) \cup Z(\tilde{h}_1) = \varnothing$, and since a Blaschke product is uniquely determined by its zeros up to a constant factor, we get $g\tilde{h}_1 = 1$. So g is invertible in $H^\infty(\Delta)$. But this is a contradiction because $g \in M_\phi$ and an invertible element cannot belong to the maximal ideal M_ϕ.

The maximal ideal M_ϕ is clearly a nonseparable, commutative Banach algebra. Also since a Banach algebra with a bounded approximate identity strongly factors [1, p. 62], our example does not have a bounded approximate identity. In fact, the results of Dixon [3, 5.10] show that it does not have an approximate identity, bounded or unbounded.

The question of whether or not a separable commutative Banach algebra that factors has factorization of pairs or of sequences remains open (see [3, §7]).

If A is an algebra and I is an ideal in A, we let

$$I^2 = \left\{ \sum_{i=1}^{n} x_i y_i : n \in \mathbb{N},\, x_1, \dots, x_n \in I,\ y_1, \dots, y_n \in I \right\} \tag{10}$$

and

$$I^{[2]} = \{xy : x \in I,\ y \in I\}.$$

In the recent paper [10], Willis gives a separable, commutative Banach algebra A such that $A^{[2]} \subsetneq A^2 = A$. We shall next use the one point part off the Šilov boundary of $H^\infty(\Delta)$, of Theorem 4.3, to give a nonseparable example of such a commutative Banach algebra.

Theorem 4.4. *There exists a nonseparable commutative Banach algebra A such that $A = A^2$ but such that A does not factor.*

Proof. Let ϕ be a one point part off the Šilov boundary of $H^\infty(\Delta)$ such that pairs do not factor in the maximal ideal M_ϕ. Let f and g be two elements of M_ϕ such that f and g have no common factor. Let $A = M_\phi \oplus M_\phi \oplus M_\phi$. With pointwise addition and scalar multiplication and the norm

$$\|(a, b, c)\| = \|a\| + \|b\| + \|c\| \qquad ((a, b, c) \in A), \tag{11}$$

A is a Banach space. Also by defining the product

$$(a_1, b_1, c_1)(a_2, b_2, c_2) = (a_1a_2,\, a_1b_2 + a_2b_1,\, a_1c_2 + a_2c_1) \tag{12}$$

for $(a_1, b_1, c_1), (a_2, b_2, c_2) \in A$, A becomes a commutative Banach algebra.

Let $(a, b, c) \in A$. Since M_ϕ factors, there exist k, l, m, n, p, and $q \in M_\phi$ such that $a = kl$, $b = mn$ and $c = pq$. Thus

$$(a, b, c) = (k, 0, 0)(l, 0, 0) + (m, 0, 0)(0, n, 0) + (p, 0, 0)(0, 0, q)$$

and so $A = A^2$. We claim that the element $(0, f, g)$, where f and g do not have a common factor in M_ϕ, does not factor. Suppose that $(0, f, g) = (a_1, b_1, c_1)(a_2, b_2, c_2)$ with $a_i, b_i, c_i \in M_\phi$ for $i = 1, 2$. Then we have

$$0 = a_1a_2, \qquad f = a_1b_2 + a_2b_1 \quad \text{and} \quad g = a_1c_2 + a_2c_1.$$

Since M_ϕ is an integral domain $0 = a_1a_2$ gives that either $a_1 = 0$ or $a_2 = 0$. It follows that either $f = a_1b_2$ and $g = a_1c_2$ or $f = b_1a_2$ and $g = c_1a_2$. This is a contradiction because f and g do not have a common factor.

Clearly this example is nonseparable.

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