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THE CYCLIC HOMOLOGY OF ALGEBRAS WITH ADJOINED UNIT

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ABSTRACT. The effect on the cyclic homology of an algebra of adjoining a unit is calculated by means of an elementary analysis at the level of chains.

1. INTRODUCTION

Let $\mathscr{A}$ be an algebra over a field k of characteristic zero. Let $\mathscr{A}^\sim$ denote the algebra over k obtained by adjoining a unit to $\mathscr{A}$. One has a split short exact sequence

$$(1) \qquad 0 \to \mathscr{A} \xrightarrow{i} \mathscr{A}^\sim \overset{p}{\rightleftarrows} k \to 0,$$

where the lifting $k \to \mathscr{A}^\sim$ maps a scalar into the corresponding multiple of the unit 1 in $\mathscr{A}^\sim$. By Propositions 4.1 and 4.2 of [2], (1) gives a split short exact sequence in cyclic homology,

$$(2) \qquad 0 \to H_n^\lambda(\mathscr{A}) \xrightarrow{i_*} H_n^\lambda(\mathscr{A}^\sim) \overset{p_*}{\rightleftarrows} H_n^\lambda(k) \to 0,$$

so that, in particular, one has

$$(3) \qquad H_n^\lambda(\mathscr{A}^\sim) \cong H_n^\lambda(\mathscr{A}) \oplus H_n^\lambda(k).$$

The purpose of this note is to give a description of this isomorphism at the level of chains. (We note that the arguments of [2] are valid also in the case of nonzero characteristic; we do not attempt to deal with this case.)

For each x in $\mathscr{A}^\sim$ set $q(x) = x - p(x) \cdot 1$. Note that q is a quasi-homomorphism from $\mathscr{A}^\sim$ onto $\mathscr{A}$. We show that q induces a map q_* : $H_n^\lambda(\mathscr{A}^\sim) \to H_n^\lambda(\mathscr{A})$ and that $q_* \oplus p_*$ gives the isomorphism in (3).

2. NOTATION AND A PRELIMINARY RESULT

We give a brief introduction to the cyclic homology of A. Connes [1]. With $\mathscr{B}$ an algebra over a field k of characteristic zero, let $\mathscr{B}^{\otimes(n+1)}$ denote the $(n+1)$-fold tensor product of $\mathscr{B}$ with itself. The Hochschild boundary operator

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$b : \mathscr{B}^{\otimes(n+1)} \to \mathscr{B}^{\otimes n}$ and the reduced boundary operator $b' : \mathscr{B}^{\otimes(n+1)} \to \mathscr{B}^{\otimes n}$ are defined by

$$b(a_0 \otimes \cdots \otimes a_n) = \sum_{j=0}^{n-1} (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1},$$

$$b'(a_0 \otimes \cdots \otimes a_n) = \sum_{j=0}^{n-1} (-1)^j a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n.$$

Define the cyclic permutation $\lambda : \mathscr{B}^{\otimes(n+1)} \to \mathscr{B}^{\otimes(n+1)}$ and the normalizing operator $A : \mathscr{B}^{\otimes(n+1)} \to \mathscr{B}^{\otimes(n+1)}$ by

$$\lambda(a_0 \otimes \cdots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \cdots \otimes a_{n-1},$$
$$A = 1 + \lambda + \lambda^2 + \cdots + \lambda^n.$$

Two n-chains (i.e. elements of $\mathscr{B}^{\otimes(n+1)}$) x and y are said to be cyclically equal if $x - y \in \operatorname{Im}(1-\lambda)$, and we write $x \equiv y$. Equivalently, $x \equiv y$ if and only if $Ax = Ay$. The formula

$$b'A = Ab \tag{4}$$

implies that b respects cyclic identifications. Let $Z_n^\lambda(\mathscr{B})$ be the set of all n-chains x with $bx \equiv 0$, and let $B_n^\lambda(\mathscr{B})$ be the cyclic image of $\mathscr{B}^{\otimes(n+2)}$ under b, so that $x \in B_n^\lambda(\mathscr{B})$ if and only if $x \equiv bz$ for some z in $\mathscr{B}^{\otimes(n+2)}$. The cyclic homology, $H_n^\lambda(\mathscr{B})$, of $\mathscr{B}$ is $Z_n^\lambda(\mathscr{B})/B_n^\lambda(\mathscr{B})$.

A linear map $\phi : \mathscr{A} \to \mathscr{B}$ gives a linear map $\phi : \mathscr{A}^{\otimes(n+1)} \to \mathscr{B}^{\otimes(n+1)}$ defined by

$$\phi(a_0 \otimes \cdots \otimes a_n) = \phi(a_0) \otimes \cdots \otimes \phi(a_n), \qquad a_j \in \mathscr{A}.$$

If ϕ is an algebra homomorphism, then ϕ induces a homomorphism ϕ_*: $H_n^\lambda(\mathscr{A}) \to H_n^\lambda(\mathscr{B})$. If ϕ is a quasi-homomorphism, that is, a difference between two homomorphisms, then ϕ does not in general, at least by this specific process, induce a map on cyclic homology.

Proposition 1. *Let* $q(x) = x - p(x) \cdot 1$, $x \in \mathscr{A}^\sim$, *be the canonical quasi-homomorphism from* $\mathscr{A}^\sim$ *onto* $\mathscr{A}$. *Then* q *induces a homomorphism* q_*: $H_n^\lambda(\mathscr{A}^\sim) \to H_n^\lambda(\mathscr{A})$, *and* $q_* i_* = \mathrm{id}\,|H_n^\lambda(\mathscr{A})$, *where* $i : \mathscr{A} \to \mathscr{A}^\sim$ *is the inclusion mapping.*

Proof. We show that

$$bqx \equiv qbx, \qquad x \in (\mathscr{A}^\sim)^{\otimes n+1}, \tag{5}$$

from which it will be clear that q maps $Z_n^\lambda(\mathscr{A}^\sim)$ into $Z_n^\lambda(\mathscr{A})$ and $B_n^\lambda(\mathscr{A}^\sim)$ into $B_n^\lambda(\mathscr{A})$. Hence $q_* : H_n^\lambda(\mathscr{A}^\sim) \to H_n^\lambda(\mathscr{A})$ is well defined and it is obvious that $q_* i_*$ is the identity on $H_n^\lambda(\mathscr{A})$.

To show (5), set $z = x - qx$ so that $qz = 0$. Write z as a linear combination of simple tensors $c_0 \otimes \cdots \otimes c_n$ with either $c_j \in \mathscr{A}$ or $c_j = 1$. We may assume that $c_j = 1$ for at least one j (otherwise $q(c_0 \otimes \cdots \otimes c_n) = c_0 \otimes \cdots \otimes c_n$). For any such j,

$$\begin{aligned} c_0 \otimes \cdots \otimes c_n &\equiv (-1)^{n(n-j+1)} c_j \otimes \cdots \otimes c_n \otimes c_0 \otimes \cdots \otimes c_{j-1} \\ &= (-1)^{n(n-j+1)} 1 \otimes c_{j+1} \otimes \cdots \otimes c_n \otimes c_0 \otimes \cdots \otimes c_{j-1}, \end{aligned}$$

and it follows that $z \equiv 1 \otimes z_0$ for some z_0 in $(\mathscr{A}^\sim)^{\otimes n}$. Thus

$$\begin{aligned} bx - bqx = bz \equiv b(1 \otimes z_0) &= (1-\lambda) z_0 - 1 \otimes b' z_0 \\ &\equiv -1 \otimes b' z_0, \end{aligned}$$

whence

$$qbx - qbqx \equiv 0. \tag{6}$$

Since $bqx \in \mathscr{A}^{\otimes(n+1)}$, $qbqx = bqx$ and (5) follows from (6). □

Note that Proposition 1 shows that the sequence (2) is exact at $H_n^\lambda(\mathscr{A})$. Exactness at $H_n^\lambda(k)$ follows from functoriality of cyclic homology. Also, we get a surjective homomorphism

$$q_* \oplus p_* : H_n^\lambda(\mathscr{A}^\sim) \to H_n^\lambda(\mathscr{A}) \oplus H_n^\lambda(k). \tag{7}$$

In the next section we shall show that $q_* \oplus p_*$ is injective, and exactness of (2) at $H_n^\lambda(\mathscr{A}^\sim)$ will follow.

3. The main result

For each n define linear maps $F\colon (\mathscr{A}^\sim)^{\otimes(n+1)} \to k \otimes (\mathscr{A}^\sim)^{\otimes(n-1)} \otimes \mathscr{A}$ and $G\colon k \otimes (\mathscr{A}^\sim)^{\otimes n} \to (\mathscr{A}^\sim)^{\otimes n}$ by

$$\begin{aligned} F(a_0 \otimes \cdots \otimes a_n) &= p(a_0) \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes q(a_n), \qquad a_j \in \mathscr{A}^\sim, \\ G(1 \otimes a_1 \otimes \cdots \otimes a_n) &= a_1 \otimes \cdots \otimes a_n, \qquad a_j \in \mathscr{A}^\sim. \end{aligned}$$

Lemma 2. *For each* $x \in (\mathscr{A}^\sim)^{\otimes(n+1)}$,

$$Fb'Ax = b'FAx - GFAx. \tag{8}$$

Proof. By linearity of (8), it suffices to establish the lemma for a simple tensor $x = a_0 \otimes \cdots \otimes a_n$ with either $a_j \in \mathscr{A}$ or $a_j = 1$. In that case $Ax = (1 + \lambda + \cdots + \lambda^n)x$ is a sum of $n+1$ tensors $c_0 \otimes \cdots \otimes c_n$ with either $c_j \in \mathscr{A}$ or $c_j = 1$. Now

$$(Fb' - b'F + GF)(c_0 \otimes \cdots \otimes c_n) = 0$$

if either

(i) $c_0 \in \mathscr{A}$,
(ii) $c_{n-1} = c_n = 1$, or
(iii) $c_0 = 1$ and $c_1,\ c_n \in \mathscr{A}$;

and, furthermore,

(iv) $(Fb' - b'F + GF)(1 \otimes 1 \otimes c_2 \otimes \cdots \otimes c_n) = 1 \otimes c_2 \otimes \cdots \otimes c_n$, $c_n \in \mathscr{A}$,

(v) $(Fb' - b'F + GF)(1 \otimes c_2 \otimes \cdots \otimes c_n \otimes 1) = (-1)^{n-1} 1 \otimes c_2 \otimes \cdots \otimes c_n$, $c_n \in \mathscr{A}$.

To see (i) and (ii) note that $Fb'(c_0 \otimes \cdots \otimes c_n) = F(c_0 \otimes \cdots \otimes c_n) = 0$ if $c_0 \in \mathscr{A}$ or if $c_{n-1} = c_n = 1$. To see (iii) assume $c_0 = 1$ and set $c = c_1 \otimes \cdots \otimes c_n$. Then

$$b'(1 \otimes c) = c - 1 \otimes b'c.$$

If $c_1 \in \mathscr{A}$ and $c_n \in \mathscr{A}$, then $Fc = 0$, $F(1 \otimes b'c) = 1 \otimes b'c$, and $F(1 \otimes c) = 1 \otimes c$, so

$$Fb'(1 \otimes c) = -1 \otimes b'c = b'(1 \otimes c) - c = b'F(1 \otimes c) - GF(1 \otimes c).$$

To see (iv) and (v), set $c' = c_2 \otimes \cdots \otimes c_n$ and assume $c_n \in \mathscr{A}$. Then

$$b'(1 \otimes 1 \otimes c') = 1 \otimes 1 \otimes b'c,$$
$$b'(1 \otimes c' \otimes 1) = c' \otimes 1 - 1 \otimes b'c' \otimes 1 + (-1)^{n-1} 1 \otimes c'.$$

Hence

$$Fb'(1 \otimes 1 \otimes c') = 1 \otimes 1 \otimes b'c' = b'F(1 \otimes 1 \otimes c'),$$

and thus

$$(Fb' - b'F + GF)(1 \otimes 1 \otimes c') = GF(1 \otimes 1 \otimes c') = 1 \otimes c'.$$

Since $F(c' \otimes 1) = 0 = F(1 \otimes b'c' \otimes 1)$ so that

$$Fb'(1 \otimes c' \otimes 1) = (-1)^{n-1} 1 \otimes c'$$

we have

$$(Fb' - b'F + GF)(1 \otimes c' \otimes 1) = (-1)^{n-1} 1 \otimes c'.$$

Finally, since

$$\lambda(1 \otimes c_2 \otimes \cdots \otimes c_n \otimes 1) = (-1)^n 1 \otimes 1 \otimes c_2 \otimes \cdots \otimes c_n,$$

we see from (iv) and (v) that if $c_n \in \mathscr{A}$, then

$$\begin{aligned}(Fb' - b'F + GF)(1 + \lambda)(1 \otimes c_2 \otimes \cdots \otimes c_n \otimes 1) \\ = (-1)^{n-1} 1 \otimes c_2 \otimes \cdots \otimes c_n + (-1)^n 1 \otimes c_2 \otimes \cdots \otimes c_n = 0.\end{aligned}$$

This, together with (i) to (iii), shows that $(Fb' - b'F + GF)Ax = 0$. □

Define $P : (\mathscr{A}^{\sim})^{\otimes(n+1)} \to (\mathscr{A}^{\sim})^{\otimes(n+1)}$ to be

$$P = \sum_{r=0}^{n} \lambda^r F \lambda^{-r}.$$

If $c = c_0 \otimes \cdots \otimes c_n$ is a simple tensor in $(\mathscr{A}^{\sim})^{\otimes(n+1)}$ with either $c_j \in \mathscr{A}$ or $c_j = 1$, then either $Fc = c$ or $Fc = 0$, and, further, either $F\lambda^{-r}c = \lambda^{-r}c$ or $F\lambda^{-r}c = 0$. Hence $Pc = lc$ for some integer $l \geq 0$. (This integer counts the number of groups of consecutive 1's in $c_0 \otimes \cdots \otimes c_n$ if c_0 is thought of as following c_n, and provided that not all c_j are 1.) We have $Pc = 0$ if and only if all $c_j \in \mathscr{A}$ or all $c_j = 1$, equivalently, if and only if either $q(c) = c$ or $p(c) = c$. More simply put, $Pc = 0$ if and only if $pc + qc = c$.

Upon writing any chain x in $(\mathscr{A}^{\sim})^{\otimes(n+1)}$ as a linear combination of simple tensors, we see that P has spectrum contained in $\{0, 1, \ldots, n\}$, that the eigenvectors of P span $(\mathscr{A}^{\sim})^{\otimes(n+1)}$ and that

$$Px = 0 \Leftrightarrow px + qx = x. \tag{9}$$

Thus we may define a partial inverse Q to P by

$$Qx = \begin{cases} 0 & \text{if } Px = 0, \\ l^{-1}x & \text{if } Px = lx,\ l \neq 0, \end{cases} \tag{10}$$

and then

$$QPx = x \Leftrightarrow px = qx = 0. \tag{11}$$

Lemma 3. *If $x \in (\mathscr{A}^{\sim})^{\otimes(n+1)}$ and $px = qx = 0$, then x is cyclically equal to $QFAx$.*

Proof. We must show that $Ax = AQFAx$. Note first that λ commutes with P and hence with Q. Thus

$$\begin{aligned} AQFAx &= \sum_{r=0}^{n}\sum_{s=0}^{n} \lambda^r QF\lambda^s x = \sum_{r=0}^{n}\sum_{s=0}^{n} Q\lambda^r F\lambda^s x \\ &= \sum_{r'=0}^{n}\sum_{s'=0}^{n} Q\lambda^{r'}\lambda^{s'} F\lambda^{-s'} x = QAPx \\ &= AQPx = Ax. \quad \square \end{aligned}$$

Define $b_0 : k \otimes (\mathscr{A}^{\sim})^{\otimes n} \to k \otimes (\mathscr{A}^{\sim})^{\otimes(n+1)}$ by

$$b_0(1 \otimes a) = -1 \otimes b'a, \qquad a \in (\mathscr{A}^{\sim})^{\otimes n}.$$

For each y in $k \otimes (\mathscr{A}^{\sim})^{\otimes n}$, the following formulas are easily verified:

$$b_0 y = b'y - Gy, \tag{12}$$

$$b(1 \otimes y) = -1 \otimes b_0 y - \lambda y. \tag{13}$$

Lemma 4. *For each z in the image of F we have*

$$b_0Pz = Pb_0z \quad \textit{and} \quad b_0Qz = Qb_0z.$$

Proof. From the definition of Q it follows that each operator that commutes with P will also commute with Q, and so it suffices to show that $b_0Pz = Pb_0z$. Secondly, it will suffice to show $b_0Pz = Pb_0z$ for each simple tensor $z = c_0 \otimes \cdots \otimes c_n$, in the image of F, with either $c_j \in \mathscr{A}$ or $c_j = 1$. We must have $c_0 = 1$ and $c_n \in \mathscr{A}$, and so we may write z as

$$z = 1 \otimes c^{(1)} \otimes 1 \otimes c^{(2)} \otimes \cdots \otimes 1 \otimes c^{(l)},$$

where

$$c^{(j)} = \underbrace{1 \otimes \cdots \otimes 1}_{r_j} \otimes c^{(j)}_{r_j+1} \otimes \cdots \otimes c^{(j)}_{s_j}, \qquad j = 1, \ldots, l,$$

$c^{(j)}_r \in \mathscr{A}$ for $r > r_j$, and $s_j > r_j \geq 0$ for $j = 1, \ldots, l$. It is easy to see that $Pz = lz$. Moreover,

$$b_0 z = -\sum_{j=1}^{l} \pm 1 \otimes c^{(1)} \otimes \cdots \otimes 1 \otimes b' c^{(j)} \otimes \cdots \otimes 1 \otimes c^{(l)}. \tag{14}$$

From the structure of $c^{(j)}$ we see that each term on the right hand side of (14) has exactly l groups of consecutive 1's (in the sense described below the proof of Lemma 2), and we conclude that $Pb_0 z = lb_0 z$. Hence $b_0 Pz = lb_0 z = Pb_0 z$ as desired. □

Lemma 5. *If* $x \in Z^\lambda_n(\mathscr{A}^\sim)$ *and* $p(x) = q(x) = 0$, *then*

$$x \equiv -b(1 \otimes QFAx),$$

so $x \in B^\lambda_n(\mathscr{A}^\sim)$.

Proof. By (13), Lemma 4, (12), Lemma 2, (4), and the hypothesis $Abx = 0$, we have

$$\begin{aligned} -b(1 \otimes QFAx) &= 1 \otimes b_0 QFAx + \lambda QFAx \\ &= 1 \otimes Qb'FAx - 1 \otimes QGFAx + \lambda QFAx \\ &= 1 \otimes QFb'Ax + \lambda QFAx \\ &= 1 \otimes QFAbx + \lambda QFAx \\ &= \lambda QFAx. \end{aligned}$$

By Lemma 3, we have

$$\lambda QFAx \equiv QFAx \equiv x. \quad \square$$

Theorem 6. *The map*

$$q_* \oplus p_* : H^\lambda_n(\mathscr{A}^\sim) \to H^\lambda_n(\mathscr{A}) \oplus H^\lambda_n(k)$$

is an isomorphism.

Proof. In Proposition 1 we saw that $q_* \oplus p_*$ is well defined and surjective. To see that $q_* \oplus p_*$ is injective, let $x \in Z^\lambda_n(\mathscr{A}^\sim)$ and set $z = x - p(x) - q(x)$. Since $p(x) \in Z^\lambda_n(k)$ and $q(x) \in Z^\lambda_n(\mathscr{A})$ (by Proposition 1) we have $z \in Z^\lambda_n(\mathscr{A}^\sim)$ and $p(z) = q(z) = 0$. From Lemma 5 it follows that $z \in B^\lambda_n(\mathscr{A}^\sim)$, that is, $[z] = 0$, so if $p_*[x] = q_*[x] = 0$, then $[x] = 0$. □

As noted in the remarks after Proposition 1, Theorem 6 implies

Corollary 7 [2]. *The sequence*

$$0 \to H_n^\lambda(\mathscr{A}) \to H_n^\lambda(\mathscr{A}^\sim) \leftrightarrows H_n^\lambda(k) \to 0$$

is exact.

We remark that Corollary 7 can be used to extend Connes's Chern character $\mathrm{ch}\colon K_0(\mathscr{A}) \to H_{2n}^\lambda(\mathscr{A})$ for unital algebras to include nonunital algebras. Indeed, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0(\mathscr{A}) & \longrightarrow & K_0(\mathscr{A}^\sim) & \longrightarrow & K_0(k) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} & & \\ 0 & \longrightarrow & H_{2n}^\lambda(\mathscr{A}) & \longrightarrow & H_{2n}^\lambda(\mathscr{A}^\sim) & \longrightarrow & H_{2n}^\lambda(k) & \longrightarrow & 0 \end{array}$$

that induces $\mathrm{ch}\colon K_0(\mathscr{A}) \to H_{2n}^\lambda(\mathscr{A})$. Note that Connes's construction of the Chern character in the odd dimensional case is already defined for nonunital algebras.

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