

MONOTONICITY OF THE FORCING TERM AND EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF SEMILINEAR ELLIPTIC PROBLEMS

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ABSTRACT. We study the existence of positive solutions to the equation $\Delta u + f(u) + \lambda g(\|x\|) = 0$ in the unit ball in \mathbb{R}^N with Dirichlet boundary conditions, where f is superlinear with $f(0) = 0$ and λ is a real parameter. We prove that if g is monotonically increasing, then there exists an $\alpha < 0$ such that for $\lambda < \alpha$ the above equation has no positive solution. This is in contrast to the case of g monotonically decreasing, where positive solutions exist for all negative values of λ .

INTRODUCTION

Let $g: [0, 1] \rightarrow (0, \infty)$ be a continuous, nondecreasing function and $\lambda \in \mathbb{R}$. Here we consider the existence of positive solutions to the equation

$$(1.1) \quad \begin{aligned} \Delta u + f(u) + \lambda g(\|x\|) &= 0 && \text{in } B_1, \\ u &= 0 && \text{on } \partial B_1, \end{aligned}$$

as λ varies over \mathbb{R} , where B_1 denotes the open unit ball in \mathbb{R}^N centered at the origin and $f: [0, \infty) \rightarrow [0, \infty)$ is differentiable, satisfying

$$(1.2) \quad f(0) = 0, \quad f''(t) \geq 0 \quad \text{with} \quad \lim_{t \rightarrow \infty} (f(t)/t) = \infty.$$

We prove that for λ 's with $|\lambda|$ large (1.1) does not have positive solutions. This contrasts with the case in which g is monotonically decreasing [2]. Indeed, taking $f(t) = t^p$, with $1 < p < (N+2)/(N-2)$, u_0 the positive radial solution of

$$\begin{aligned} \Delta w + w^p &= 0 && \text{in } B_1, \\ w &= 0 && \text{on } \partial B_1, \end{aligned}$$

and then taking $g(r) = u_0^p(r)$, it is easily seen that (1.1) has a positive solution for each $\lambda \leq 0$.

Our results extend and complement the work of Ramaswamy [4] and of Brown, Castro, and Shivaji [1].

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Theorem 1. *If (1.2) holds then there exist $\alpha < 0$ and $\beta > 0$ such that, for $\lambda \notin [\alpha, \beta]$, equation (1.1) has no positive solutions.*

Proof. The existence of β follows from a simple integration-by-parts argument using the fact that f is superlinear.

Let λ be negative. From Gidas, Ni, and Nirenberg [3] it follows that any positive solution to (1.1) is radially symmetric and decreasing. Hence (1.1) is equivalent to the singular ordinary differential equation

$$(1.3) \quad \begin{aligned} u'' + \frac{n}{r}u' + f(u) - \lambda g(r) &= 0 \quad \text{in } (0, 1), \\ u'(0) &= 0, \quad u(1) = 0, \end{aligned}$$

where $\lambda > 0$ and $n = N - 1$. If we assume, on the contrary, that such an α does not exist, then there exists a sequence (u_j, λ_j) satisfying (1.3) with $\lambda_j \rightarrow \infty$. Since $u = u_j$ is radially decreasing on $[0, 1]$, we have $\Delta u \leq 0$ at 0 and $\Delta u > 0$ at 1. Thus there exists $a_j \in [0, 1)$ such that $\Delta u = 0$ at a_j . That is,

$$(1.4) \quad f(u(a_j)) = \lambda_j g(a_j).$$

Next we prove that $a_j \rightarrow 1$ as $\lambda_j \rightarrow \infty$. Since u is radially decreasing, there exists a $b_j \in (a_j, 1)$ such that

$$(1.5) \quad 2f(u(b_j)) = \lambda_j g(a_j).$$

Since g is nondecreasing, we have $a_j < b_j$. This and (1.5) imply that

$$(1.6) \quad \lambda_j g(0) \leq 2f(u(b_j)) = \lambda_j g(a_j) \leq \lambda_j g(b_j) \leq \lambda_j g(1).$$

Because of (1.6), for $t \in [b_j, 1]$, we have

$$(t^n u')' = t^n (\lambda_j g(t) - f(u(t))) \geq \frac{1}{2} t^n \lambda_j g(t),$$

using the fact that f, g are nondecreasing and u is decreasing. (Note that (1.2) forces f to be nondecreasing.) Now, integrating on $[b_j, 1]$, we get

$$u'(1) - b_j^n u'(b_j) \geq \lambda_j g(b_j) (1 - b_j^N) / (2N).$$

Thus

$$(1.7) \quad g(0)\lambda_j(1 - b_j^N) \leq 2N(b_j^n |u'(b_j)| - |u'(1)|),$$

using (1.6) and, since $u'(b_j) < 0, u'(1) \leq 0$. (Also note that our choice of a_j as in (1.4) implies that $u'' > 0$ in $(a_j, 1)$.) Now, multiplying (1.3) by $r^{2n}u'$ and integrating on $[b_j, 1]$ we obtain

$$\begin{aligned} (u'(1))^2 - b_j^{2n}(u'(b_j))^2 &\geq 2b_j^{2n}F(u(b_j)) + 4n \int_{b_j}^1 r^{2n-1}F(u) + 2\lambda_j \int_{b_j}^1 r^{2n}gu' \\ &> 2\lambda_j \int_{b_j}^1 r^{2n}gu' \geq -2\lambda_j g(1)u(b_j). \end{aligned}$$

This and (1.5) yield

$$(1.8) \quad b_j^{2n} (u'(b_j))^2 - (u'(1))^2 < cf(u(b_j))u(b_j),$$

where c is a constant independent of j . From (1.7) and (1.8), we have

$$(1 - b_j^N)^2 < c_1 u(b_j) / f(u(b_j)) \rightarrow 0$$

as $j \rightarrow \infty$ (see (1.4) and (1.5)). Hence $b_j \rightarrow 1$ as $j \rightarrow \infty$.

Now, by the convexity of $f \circ u$ in $[a_j, 1]$, we infer that $f(u((1 + a_j)/2)) \leq f(u(a_j))/2 = f(u(b_j))$. Thus $(1 + a_j) \geq 2b_j$, proving that $a_j \rightarrow 1$ as $j \rightarrow \infty$.

Now let $\varepsilon > 0$, and choose j large enough such that $1 - a_j < \varepsilon$. Let $\mu_1 = \mu_1(-\Delta, B_{1-\varepsilon})$ denote the first eigenvalue of $-\Delta$ in $B_{1-\varepsilon}$, a ball of radius $(1 - \varepsilon)$ around the origin. That is,

$$(1.9) \quad \begin{aligned} -\Delta \varphi_1 &= \mu_1 \varphi_1 && \text{in } B_{1-\varepsilon}, \\ \varphi_1 &= 0 && \text{on } \partial B_{1-\varepsilon}, \end{aligned}$$

where φ_1 is chosen to be positive in $B_{1-\varepsilon}$. We choose j large enough so that $f'(u(a_j)) > \mu_1$. This is possible by virtue of (1.2), along with (1.4) and (1.6). Then $v(x) = u(x) - u(a_j)$ satisfies

$$\begin{aligned} -\Delta v &> \mu_1(-\Delta, B_{a_j})v && \text{in } B_{a_j}, \\ v &= 0 && \text{on } \partial B_{a_j}, \end{aligned}$$

with $v > 0$ in B_{a_j} , which yields a contradiction if we consider the eigenvalue problem (1.9) in B_{a_j} . Hence the theorem is proven. \square

Remark 1. In contrast, when g is decreasing, we might have the existence of positive solutions for all $\lambda < 0$. For example, when $f(t) = t|t|^{p-1}$, $1 < p < (N+2)/(N-2)$ for $N \geq 3$, and $g(x) = u_0^p(x)$ for any $\lambda < 0$, there is a positive radial solution of (1.1) given by cu_0 , where c is such that $c - c^p = \lambda$. Thus the monotonicity properties of g and the positivity of g on the boundary ∂B_1 play an important role in the existence results.

Remark 2. It can be easily seen that there is no nonradial degeneracy all along the positive solution curve for negative λ 's as opposed to symmetry breaking in the case when g is decreasing [2].

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