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MONOTONICITY OF THE FORCING TERM AND EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF SEMILINEAR ELLIPTIC PROBLEMS

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ABSTRACT. We study the existence of positive solutions to the equation $\Delta u + f(u) + \lambda g(\|x\|) = 0$ in the unit ball in $\mathbb{R}^N$ with Dirichlet boundary conditions, where f is superlinear with $f(0) = 0$ and λ is a real parameter. We prove that if g is monotonically increasing, then there exists an $\alpha < 0$ such that for $\lambda < \alpha$ the above equation has no positive solution. This is in contrast to the case of g monotonically decreasing, where positive solutions exist for all negative values of λ.

INTRODUCTION

Let $g: [0, 1] \to (0, \infty)$ be a continuous, nondecreasing function and $\lambda \in \mathbb{R}$. Here we consider the existence of positive solutions to the equation

$$\begin{aligned} \Delta u + f(u) + \lambda g(\|x\|) &= 0 \quad \text{in } B_1, \\ u &= 0 \quad \text{on } \partial B_1, \end{aligned} \tag{1.1}$$

as λ varies over $\mathbb{R}$, where B_1 denotes the open unit ball in $\mathbb{R}^N$ centered at the origin and $f: [0, \infty) \to [0, \infty)$ is differentiable, satisfying

$$f(0) = 0, \qquad f''(t) \geq 0 \quad \text{with } \lim_{t\to\infty} (f(t)/t) = \infty. \tag{1.2}$$

We prove that for λ's with $|\lambda|$ large (1.1) does not have positive solutions. This contrasts with the case in which g is monotonically decreasing [2]. Indeed, taking $f(t) = t^p$, with $1 < p < (N+2)/(N-2)$, u_0 the positive radial solution of

$$\begin{aligned} \Delta w + w^p &= 0 \quad \text{in } B_1, \\ w &= 0 \quad \text{on } \partial B_1, \end{aligned}$$

and then taking $g(r) = u_0^p(r)$, it is easily seen that (1.1) has a positive solution for each $\lambda \leq 0$.

Our results extend and complement the work of Ramaswamy [4] and of Brown, Castro, and Shivaji [1].

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Theorem 1. *If* (1.2) *holds then there exist* $\alpha < 0$ *and* $\beta > 0$ *such that, for* $\lambda \notin [\alpha, \beta]$, *equation* (1.1) *has no positive solutions.*

Proof. The existence of β follows from a simple integration-by-parts argument using the fact that f is superlinear.

Let λ be negative. From Gidas, Ni, and Nirenberg [3] it follows that any positive solution to (1.1) is radially symmetric and decreasing. Hence (1.1) is equivalent to the singular ordinary differential equation

$$
\begin{gathered}
u'' + \frac{n}{r}u' + f(u) - \lambda g(r) = 0 \quad \text{in } (0, 1), \\
u'(0) = 0, \qquad u(1) = 0,
\end{gathered}
\tag{1.3}
$$

where $\lambda > 0$ and $n = N - 1$. If we assume, on the contrary, that such an α does not exist, then there exists a sequence (u_j, λ_j) satisfying (1.3) with $\lambda_j \to \infty$. Since $u = u_j$ is radially decreasing on $[0, 1]$, we have $\Delta u \leq 0$ at 0 and $\Delta u > 0$ at 1. Thus there exists $a_j \in [0, 1)$ such that $\Delta u = 0$ at a_j. That is,

$$
f(u(a_j)) = \lambda_j g(a_j). \tag{1.4}
$$

Next we prove that $a_j \to 1$ as $\lambda_j \to \infty$. Since u is radially decreasing, there exists a $b_j \in (a_j, 1)$ such that

$$
2f(u(b_j)) = \lambda_j g(a_j). \tag{1.5}
$$

Since g is nondecreasing, we have $a_j < b_j$. This and (1.5) imply that

$$
\lambda_j g(0) \leq 2f(u(b_j)) = \lambda_j g(a_j) \leq \lambda_j g(b_j) \leq \lambda_j g(1). \tag{1.6}
$$

Because of (1.6), for $t \in [b_j, 1]$, we have

$$
(t^n u')' = t^n(\lambda_j g(t) - f(u(t))) \geq \tfrac{1}{2} t^n \lambda_j g(t),
$$

using the fact that f, g are nondecreasing and u is decreasing. (Note that (1.2) forces f to be nondecreasing.) Now, integrating on $[b_j, 1]$, we get

$$
u'(1) - b_j^n u'(b_j) \geq \lambda_j g(b_j)(1 - b_j^N)/(2N).
$$

Thus

$$
g(0)\lambda_j(1 - b_j^N) \leq 2N(b_j^n |u'(b_j)| - |u'(1)|), \tag{1.7}
$$

using (1.6) and, since $u'(b_j) < 0$, $u'(1) \leq 0$. (Also note that our choice of a_j as in (1.4) implies that $u'' > 0$ in $(a_j, 1)$.) Now, multiplying (1.3) by $r^{2n}u'$ and integrating on $[b_j, 1]$ we obtain

$$
\begin{aligned}
(u'(1))^2 - b_j^{2n}(u'(b_j))^2 &\geq 2b_j^{2n} F(u(b_j)) + 4n \int_{b_j}^1 r^{2n-1} F(u) + 2\lambda_j \int_{b_j}^1 r^{2n} g u' \\
&> 2\lambda_j \int_{b_j}^1 r^{2n} g u' \geq -2\lambda_j g(1) u(b_j).
\end{aligned}
$$

This and (1.5) yield

$$b_j^{2n}(u'(b_j))^2 - (u'(1))^2 < cf(u(b_j))u(b_j), \tag{1.8}$$

where c is a constant independent of j. From (1.7) and (1.8), we have

$$(1 - b_j^N)^2 < c_1 u(b_j)/f(u(b_j)) \to 0$$

as $j \to \infty$ (see (1.4) and (1.5)). Hence $b_j \to 1$ as $j \to \infty$.

Now, by the convexity of $f \circ u$ in $[a_j, 1]$, we infer that $f(u((1 + a_j)/2)) \leq f(u(a_j))/2 = f(u(b_j))$. Thus $(1 + a_j) \geq 2b_j$, proving that $a_j \to 1$ as $j \to \infty$.

Now let $\varepsilon > 0$, and choose j large enough such that $1 - a_j < \varepsilon$. Let $\mu_1 = \mu_1(-\Delta, B_{1-\varepsilon})$ denote the first eigenvalue of $-\Delta$ in $B_{1-\varepsilon}$, a ball of radius $(1 - \varepsilon)$ around the origin. That is,

$$\begin{aligned} -\Delta\varphi_1 &= \mu_1\varphi_1 \quad && \text{in } B_{1-\varepsilon}, \\ \varphi_1 &= 0 && \text{on } \partial B_{1-\varepsilon}, \end{aligned} \tag{1.9}$$

where φ_1 is chosen to be positive in $B_{1-\varepsilon}$. We choose j large enough so that $f'(u(a_j)) > \mu_1$. This is possible by virtue of (1.2), along with (1.4) and (1.6). Then $v(x) = u(x) - u(a_j)$ satisfies

$$\begin{aligned} -\Delta v > \mu_1(-\Delta, B_{a_j})v \quad && \text{in } B_{a_j}, \\ v = 0 && \text{on } \partial B_{a_j}, \end{aligned}$$

with $v > 0$ in B_{a_j}, which yields a contradiction if we consider the eigenvalue problem (1.9) in B_{a_j}. Hence the theorem is proven. □

Remark 1. In contrast, when g is decreasing, we might have the existence of positive solutions for all $\lambda < 0$. For example, when $f(t) = t|t|^{p-1}$, $1 < p < (N+2)/(N-2)$ for $N \geq 3$, and $g(x) = u_0^p(x)$ for any $\lambda < 0$, there is a positive radial solution of (1.1) given by cu_0, where c is such that $c - c^p = \lambda$. Thus the monotonicity properties of g and the positivity of g on the boundary ∂B_1 play an important role in the existence results.

Remark 2. It can be easily seen that there is no nonradial degeneracy all along the positive solution curve for negative λ's as opposed to symmetry breaking in the case when g is decreasing [2].

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