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INTEGRAL MEANS, BOUNDED MEAN OSCILLATION, AND GELFER FUNCTIONS

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ABSTRACT. A Gelfer function f is a holomorphic function in the unit disc $D = \{z: |z| < 1\}$ such that $f(0) = 1$ and $f(z) + f(w) \neq 0$ for all z, w in D. The family G of Gelfer functions contains the family P of holomorphic functions f in D with $f(0) = 1$ and $\operatorname{Re} f > 0$ in D. Yamashita has recently proved that if f is a Gelfer function then $f \in H^p$, $0 < p < 1$, while $\log f \in \mathrm{BMOA}$ and $\|\log f\|_{\mathrm{BMOA}_2} \leq \pi/\sqrt{2}$. In this paper we prove that the function $\lambda(z) = (1+z)/(1-z)$ is extremal for a very large class of problems about integral means in the class G. This result in particular implies that $G \subset H^p$, $0 < p < 1$, and we use it also to obtain a new proof of a generalization of Yamashita's estimation of the BMOA norm of $\log f$, $f \in G$.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let D denote the unit disc $\{z: |z| < 1\}$. For $p > 0$ and g analytic in D, define

$$M_p(r, g) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(re^{it})|^p\, dt\right)^{1/p}, \qquad 0 < r < 1,$$

$$M_\infty(r, g) = \max_{|z|=r} |g(z)|, \qquad 0 < r < 1.$$

The Hardy space H^p consists of those g analytic in D for which

$$\|g\|_{H^p} = \sup_{0<r<1} M_p(r, g) < \infty.$$

Let G be the class of functions f analytic in D with $f(0) = 1$ and having the Gelfer property

$$f(z) + f(w) \neq 0 \quad \text{for all } z, w \in D. \tag{1}$$

The class G was introduced by Gelfer [7] and we shall call a member of G a Gelfer function. We refer to [7; 8; 5, pp. 266–267; 9, vol. II, pp. 73–76, 82–83] for the theory of Gelfer functions.

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An important subclass of G is the class P of functions f analytic in D with $f(0) = 1$ and $\operatorname{Re} f(z) > 0$ for all z in D. The function

$$\lambda(z) = (1+z)/(1-z) \tag{2}$$

is extremal for many problems in the classes P and G. It is well known that

$$P \subset \bigcap_{0<p<1} H^p$$

(see [4, p. 13]). S. Yamashita has recently proved [11, Theorem 2] that the same is true for the bigger class G. The first result in this paper asserts that the function λ is extremal for a large class of problems about integral means in the class G.

Theorem 1. *Let f be a Gelfer function. Then for each convex function Φ on $\mathbb{R}$, we have*

$$\int_{-\pi}^{\pi} \Phi(\log|f(re^{it})|)\,dt \le \int_{-\pi}^{\pi} \Phi(\log|\lambda(re^{it})|)\,dt, \qquad 0<r<1. \tag{3}$$

In particular,

$$M_p(r, f) \le M_p(r, \lambda), \qquad 0<r<1,\ 0<p\le\infty. \tag{4}$$

Hence $f \in \bigcap_{0<p<1} H^p$ and

$$\|f\|_{H^p} \le \|\lambda\|_{H^p}, \qquad 0<p<1. \tag{5}$$

Let BMOA be the space of functions f in H^1 whose boundary values have bounded mean oscillation on ∂D. There are many characterizations of BMOA-functions (see [2, 6, 10]). We are interested in the following.

Let $0<p<\infty$. A function g analytic in D is in BMOA if and only if

$$\|g\|_{\mathrm{BMOA}_p} = |g(0)| + \sup_{w\in D} \|g_w\|_{H^p} < \infty,$$

where

$$f_w(z) = f\left(\frac{z+w}{1+\overline{w}z}\right) - f(w).$$

Yamashita [11, Theorem 3] has proved that if f is a Gelfer function then $\log f \in \mathrm{BMOA}$ and

$$\|\log f\|_{\mathrm{BMOA}_2} \le \|\log \lambda\|_{\mathrm{BMOA}_2} = \pi/\sqrt{2}. \tag{6}$$

Yamashita's proof of (6) is based on the fact that if f is a univalent Gelfer function then f^2 is univalent and hence Theorems 4 and 5 of [8] together with Danikas' computation of the BMOA_2-norm of $\log(1-z)$ [3] imply (6) for any univalent Gelfer function. The result for a general Gelfer function follows from the fact that a Gelfer function is subordinate to a univalent Gelfer function and that subordination decreases the BMOA_2-norm. Let us notice that this argument works if we consider the BMOA_p-norms, $0<p\le 2$. We have:

Let f be a Gelfer function. Then

$$\|\log f\|_{\mathrm{BMOA}_p} \leq \|\log \lambda\|_{\mathrm{BMOA}_p}, \qquad 0 < p \leq 2. \tag{7}$$

We shall give a new proof of this result based on Theorem 1 and the results of [8]. This proof applies directly to an arbitrary Gelfer function; it does not need to use the fact that a Gelfer function is subordinate to a univalent Gelfer function.

2. Proofs of the results

The proof of Theorem 1 is based on the following elementary observation: Let f be a Gelfer function. For $0 < r < \infty$, let

$$R(r) = \{t \in [-\pi, \pi]: \mathrm{r}e^{it} \in f(D)\}$$

and let $m(r)$ denote the Lebesgue measure of $R(r)$. Then the Gelfer condition (1) implies

$$m(r) \leq \pi, \qquad 0 < r < \infty. \tag{8}$$

Proof of Theorem 1. Let f be a Gelfer function and set $R = f(D)$. Let R^* be the circular symmetrization of R (see [1, pp. 141–142]). Then (8) implies

$$R^* \subset \{z: \operatorname{Re} z > 0\}, \tag{9}$$

and then, since $\lambda(z) = (1+z)/(1-z)$ is a conformal mapping of D onto $\{z: \operatorname{Re} z > 0\}$ with $\lambda(0) = f(0) = 1$, Theorem 6 of [1] implies (3) for every convex increasing function Φ on $\mathbb{R}$.

Now, it is a simple exercise to show that a convex function on $\mathbb{R}$ can be written as the sum of a convex increasing function on $\mathbb{R}$ and a convex decreasing function on $\mathbb{R}$. Therefore we need to show that (3) holds for every convex decreasing function Φ on $\mathbb{R}$. To prove this observe that if $f \in G$ then $1/f \in G$ and, hence

$$\int_{-\pi}^{\pi} \Psi(-\log|f(re^{it})|)\, dt \leq \int_{-\pi}^{\pi} \Psi(\log|\lambda(re^{it})|)\, dt, \qquad 0 < r < 1,$$

for every convex increasing function Ψ on R. Notice that $|\lambda(re^{it})|$ and $1/|\lambda(re^{it})|$ are equidistributed on $[-\pi, \pi]$ and hence

$$\int_{-\pi}^{\pi} \Psi(\log|\lambda(re^{it})|)\, dt = \int_{-\pi}^{\pi} \Psi(-\log|\lambda(re^{it})|\, dt.$$

Consequently, we have

$$\int_{-\pi}^{\pi} \Psi(-\log|f(re^{it})|)\, dt \leq \int_{-\pi}^{\pi} |\Psi(-\log|\lambda(re^{it})|)\, dt, \qquad 0 < r < 1, \tag{10}$$

for every convex increasing function Ψ on $\mathbb{R}$.

Now, if $\Phi(x)$ is a convex decreasing function on $\mathbb{R}$ then $\Phi(-x)$ is a convex increasing function on $\mathbb{R}$ and then (10) shows that

$$\int_{-\pi}^{\pi} \Phi(\log|f(re^{it})|)\, dt \leq \int_{-\pi}^{\pi} \Phi(\log|\lambda(re^{it})|)\, dt, \qquad 0 < r < 1.$$

This finishes the proof of (3) for any convex function Φ on $\mathbb{R}$.

The other assertions of Theorem 1 follow easily from (3).

We turn now to study the integral means of $\log f$, $f \in G$. First of all let us notice that if $f \in P$ then $\log f$ is subordinate to $\log \lambda$ and the Littlewood subordination theorem [4, p. 10] shows that, for $0 < p \leq \infty$,

$$M_p(r, \log f) \leq M_p(r, \log \lambda), \qquad 0 < r < 1.$$

In Theorem 2 we shall prove that this is true for $f \in G$ and $0 < p \leq 2$ and we shall see that (7) follows easily from this result.

Theorem 2. *Let f be a Gelfer function. Then, for $0 < p \leq 2$,*

$$M_p(r, \log f) \leq M_p(r, \log \lambda), \qquad 0 < r < 1, \tag{11}$$

and

$$\| \log f \|_{\mathrm{BMOA}_p} \leq \| \log \lambda \|_{\mathrm{BMOA}_p}. \tag{12}$$

Proof. First let us notice that, for $p = 2$, (11) follows trivially from Theorem 1. In fact, if we take $\Phi(x) = x^2$ in (3), we obtain

$$M_2(r, \operatorname{Re} \log f) \leq M_2(r, \operatorname{Re} \log \lambda), \qquad 0 < r < 1,$$

which, with Parseval equality [4, p. 54], gives

$$M_2(r, \log f) \leq M_2(r, \log \lambda), \qquad 0 < r < 1.$$

In order to obtain (11) for $0 < p < 2$, we use the methods and results of [8]. Recall [1] that if u is a real valued function defined in D such that, for $0 < r < 1$, $u(re^{it})$ is integrable on $[-\pi, \pi]$, the function u^* is defined in $D^+ = D \cap \{\operatorname{Im} z > 0\}$ by

$$u^*(re^{it}) = \sup_{|E|=2t} \int_E u(re^{is})\, ds, \qquad 0 < r < 1, \ 0 < t < \pi,$$

where $|E|$ denotes the Lebesgue measure of E.

Let $f \in G$ and $0 < r < 1$. Set

$$g(z) = \log f(rz), \qquad z \in \overline{D}, \tag{13}$$

$$h(z) = \log \lambda(rz), \qquad z \in \overline{D}. \tag{14}$$

Then g and h are analytic in $\overline{D}$ and $g(0) = h(0) = 0$. According to [1, Proposition 3], the inequality (3) for every convex increasing function Φ on $\mathbb{R}$ implies

$$(\operatorname{Re} g)^* \leq (\operatorname{Re} h)^* \quad \textit{in } D^+, \tag{15}$$

while (3) for every convex decreasing function Φ on R gives

$$(-\operatorname{Re} g)^* \leq (-\operatorname{Re} h)^* \quad \textit{in } D^+. \tag{16}$$

Notice that (15) implies

$$\max_{z\in D} \operatorname{Re} g(z) \le \max_{z\in D} \operatorname{Re} h(z) \tag{17}$$

and (16) implies

$$\min_{z\in D} \operatorname{Re} h(z) \le \min_{z\in D} \operatorname{Re} g(z). \tag{18}$$

Moreover, h is univalent and an argument like that used in the proof of Lemma 1 of [8] shows that $h(D)$ is a Steiner symmetric domain.

Hence, Proposition 6 of [8] implies

$$\int_{-\pi}^{\pi} |g(e^{it})|^p \, dt \le \int_{-\pi}^{\pi} |h(e^{it})|^p \, dt, \qquad 0 < p \le 2,$$

which is equivalent to

$$M_p(r, \log f) \le M_p(r, \log \lambda), \qquad 0 < p \le 2.$$

To prove (12), observe that (11) implies that if f is a Gelfer function then

$$\|\log f\|_{H^p} \le \|\log \lambda\|_{H^p}, \qquad 0 < p \le 2. \tag{19}$$

Now, if $f \in G$, $w \in D$, and

$$g(z) = f\left(\frac{z+w}{1+\overline{w}z}\right) \Big/ f(w),$$

then $g \in G$ and hence

$$\|\log g\|_{H^p} \le \|\log \lambda\|_{H^p}, \qquad 0 < p \le 2. \tag{20}$$

But notice that

$$\log g = (\log f)_w$$

and then (20) implies that, for $0 < p \le 2$,

$$\|\log f\|_{H^p} = \sup_{w\in D} \|(\log f)_w\|_{H^p} \le \|\log \lambda\|_{H^p} = \|\log \lambda\|_{\mathrm{BMOA}_p}.$$

This proves (12), finishing the proof of Theorem 2.

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