

INTEGRAL MEANS, BOUNDED MEAN OSCILLATION, AND GELFER FUNCTIONS

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ABSTRACT. A Gelfer function f is a holomorphic function in the unit disc $D = \{z: |z| < 1\}$ such that $f(0) = 1$ and $f(z) + f(w) \neq 0$ for all z, w in D . The family G of Gelfer functions contains the family P of holomorphic functions f in D with $f(0) = 1$ and $\operatorname{Re} f > 0$ in D . Yamashita has recently proved that if f is a Gelfer function then $f \in H^p$, $0 < p < 1$, while $\log f \in \text{BMOA}$ and $\|\log f\|_{\text{BMOA}_2} \leq \pi/\sqrt{2}$. In this paper we prove that the function $\lambda(z) = (1+z)/(1-z)$ is extremal for a very large class of problems about integral means in the class G . This result in particular implies that $G \subset H^p$, $0 < p < 1$, and we use it also to obtain a new proof of a generalization of Yamashita's estimation of the BMOA norm of $\log f$, $f \in G$.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let D denote the unit disc $\{z: |z| < 1\}$. For $p > 0$ and g analytic in D , define

$$M_p(r, g) = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |g(re^{it})|^p dt \right)^{1/p}, \quad 0 < r < 1,$$

$$M_\infty(r, g) = \max_{|z|=r} |g(z)|, \quad 0 < r < 1.$$

The Hardy space H^p consists of those g analytic in D for which

$$\|g\|_{H^p} = \sup_{0 < r < 1} M_p(r, g) < \infty.$$

Let G be the class of functions f analytic in D with $f(0) = 1$ and having the Gelfer property

$$(1) \quad f(z) + f(w) \neq 0 \quad \text{for all } z, w \in D.$$

The class G was introduced by Gelfer [7] and we shall call a member of G a Gelfer function. We refer to [7; 8; 5, pp. 266–267; 9, vol. II, pp. 73–76, 82–83] for the theory of Gelfer functions.

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An important subclass of G is the class P of functions f analytic in D with $f(0) = 1$ and $\operatorname{Re} f(z) > 0$ for all z in D . The function

$$(2) \quad \lambda(z) = (1 + z)/(1 - z)$$

is extremal for many problems in the classes P and G . It is well known that

$$P \subset \bigcap_{0 < p < 1} H^p$$

(see [4, p. 13]). S. Yamashita has recently proved [11, Theorem 2] that the same is true for the bigger class G . The first result in this paper asserts that the function λ is extremal for a large class of problems about integral means in the class G .

Theorem 1. *Let f be a Gelfer function. Then for each convex function Φ on \mathbb{R} , we have*

$$(3) \quad \int_{-\pi}^{\pi} \Phi(\log |f(re^{it})|) dt \leq \int_{-\pi}^{\pi} \Phi(\log |\lambda(re^{it})|) dt, \quad 0 < r < 1.$$

In particular,

$$(4) \quad M_p(r, f) \leq M_p(r, \lambda), \quad 0 < r < 1, \quad 0 < p \leq \infty.$$

Hence $f \in \bigcap_{0 < p < 1} H^p$ and

$$(5) \quad \|f\|_{H^p} \leq \|\lambda\|_{H^p}, \quad 0 < p < 1.$$

Let $BMOA$ be the space of functions f in H^1 whose boundary values have bounded mean oscillation on ∂D . There are many characterizations of $BMOA$ -functions (see [2, 6, 10]). We are interested in the following.

Let $0 < p < \infty$. A function g analytic in D is in $BMOA$ if and only if

$$\|\delta\|_{BMOA_p} = |g(0)| + \sup_{w \in D} \|g_w\|_{H^p} < \infty,$$

where

$$f_w(z) = f\left(\frac{z+w}{1+\bar{w}z}\right) - f(w).$$

Yamashita [11, Theorem 3] has proved that if f is a Gelfer function then $\log f \in BMOA$ and

$$(6) \quad \|\log f\|_{BMOA_2} \leq \|\log \lambda\|_{BMOA_2} = \pi/\sqrt{2}.$$

Yamashita's proof of (6) is based on the fact that if f is a univalent Gelfer function then f^2 is univalent and hence Theorems 4 and 5 of [8] together with Danikas' computation of the $BMOA_2$ -norm of $\log(1 - z)$ [3] imply (6) for any univalent Gelfer function. The result for a general Gelfer function follows from the fact that a Gelfer function is subordinate to a univalent Gelfer function and that subordination decreases the $BMOA_2$ -norm. Let us notice that this argument works if we consider the $BMOA_p$ -norms, $0 < p \leq 2$. We have:

Let f be a Gelfer function. Then

$$(7) \quad \|\log f\|_{\text{BMOA}_p} \leq \|\log \lambda\|_{\text{BMOA}_p}, \quad 0 < p \leq 2.$$

We shall give a new proof of this result based on Theorem 1 and the results of [8]. This proof applies directly to an arbitrary Gelfer function; it does not need to use the fact that a Gelfer function is subordinate to a univalent Gelfer function.

2. PROOFS OF THE RESULTS

The proof of Theorem 1 is based on the following elementary observation:

Let f be a Gelfer function. For $0 < r < \infty$, let

$$R(r) = \{t \in [-\pi, \pi]: re^{it} \in f(D)\}$$

and let $m(r)$ denote the Lebesgue measure of $R(r)$. Then the Gelfer condition (1) implies

$$(8) \quad m(r) \leq \pi, \quad 0 < r < \infty.$$

Proof of Theorem 1. Let f be a Gelfer function and set $R = f(D)$. Let R^* be the circular symmetrization of R (see [1, pp. 141–142]). Then (8) implies

$$(9) \quad R^* \subset \{z: \text{Re } z > 0\},$$

and then, since $\lambda(z) = (1+z)/(1-z)$ is a conformal mapping of D onto $\{z: \text{Re } z > 0\}$ with $\lambda(0) = f(0) = 1$, Theorem 6 of [1] implies (3) for every convex increasing function Φ on \mathbb{R} .

Now, it is a simple exercise to show that a convex function on \mathbb{R} can be written as the sum of a convex increasing function on \mathbb{R} and a convex decreasing function on \mathbb{R} . Therefore we need to show that (3) holds for every convex decreasing function Φ on \mathbb{R} . To prove this observe that if $f \in G$ then $1/f \in G$ and, hence

$$\int_{-\pi}^{\pi} \Psi(-\log |f(re^{it})|) dt \leq \int_{-\pi}^{\pi} \Psi(\log |\lambda(re^{it})|) dt, \quad 0 < r < 1,$$

for every convex increasing function Ψ on \mathbb{R} . Notice that $|\lambda(re^{it})|$ and $1/|f(re^{it})|$ are equidistributed on $[-\pi, \pi]$ and hence

$$\int_{-\pi}^{\pi} \Psi(\log |\lambda(re^{it})|) dt = \int_{-\pi}^{\pi} \Psi(-\log |f(re^{it})|) dt.$$

Consequently, we have

$$(10) \quad \int_{-\pi}^{\pi} \Psi(-\log |f(re^{it})|) dt \leq \int_{-\pi}^{\pi} |\Psi(-\log |\lambda(re^{it})|)| dt, \quad 0 < r < 1,$$

for every convex increasing function Ψ on \mathbb{R} .

Now, if $\Phi(x)$ is a convex decreasing function on \mathbb{R} then $\Phi(-x)$ is a convex increasing function on \mathbb{R} and then (10) shows that

$$\int_{-\pi}^{\pi} \Phi(\log |f(re^{it})|) dt \leq \int_{-\pi}^{\pi} \Phi(\log |\lambda(re^{it})|) dt, \quad 0 < r < 1.$$

This finishes the proof of (3) for any convex function Φ on \mathbb{R} .

The other assertions of Theorem 1 follow easily from (3).

We turn now to study the integral means of $\log f$, $f \in G$. First of all let us notice that if $f \in P$ then $\log f$ is subordinate to $\log \lambda$ and the Littlewood subordination theorem [4, p. 10] shows that, for $0 < p \leq \infty$,

$$M_p(r, \log f) \leq M_p(r, \log \lambda), \quad 0 < r < 1.$$

In Theorem 2 we shall prove that this is true for $f \in G$ and $0 < p \leq 2$ and we shall see that (7) follows easily from this result.

Theorem 2. *Let f be a Gelfer function. Then, for $0 < p \leq 2$,*

$$(11) \quad M_p(r, \log f) \leq M_p(r, \log \lambda), \quad 0 < r < 1,$$

and

$$(12) \quad \|\log f\|_{\text{BMOA}_p} \leq \|\log \lambda\|_{\text{BMOA}_p}.$$

Proof. First let us notice that, for $p = 2$, (11) follows trivially from Theorem 1. In fact, if we take $\Phi(x) = x^2$ in (3), we obtain

$$M_2(r, \text{Re } \log f) \leq M_2(r, \text{Re } \log \lambda), \quad 0 < r < 1,$$

which, with Parseval equality [4, p. 54], gives

$$M_2(r, \log f) \leq M_2(r, \log \lambda), \quad 0 < r < 1.$$

In order to obtain (11) for $0 < p < 2$, we use the methods and results of [8]. Recall [1] that if u is a real valued function defined in D such that, for $0 < r < 1$, $u(re^{it})$ is integrable on $[-\pi, \pi]$, the function u^* is defined in $D^+ = D \cap \{\text{Im } z > 0\}$ by

$$u^*(re^{it}) = \sup_{|E|=2t} \int_E u(re^{is}) ds, \quad 0 < r < 1, \quad 0 < t < \pi,$$

where $|E|$ denotes the Lebesgue measure of E .

Let $f \in G$ and $0 < r < 1$. Set

$$(13) \quad g(z) = \log f(rz), \quad z \in \bar{D},$$

$$(14) \quad h(z) = \log \lambda(rz), \quad z \in \bar{D}.$$

Then g and h are analytic in \bar{D} and $g(0) = h(0) = 0$. According to [1, Proposition 3], the inequality (3) for every convex increasing function Φ on \mathbb{R} implies

$$(15) \quad (\text{Re } g)^* \leq (\text{Re } h)^* \quad \text{in } D^+,$$

while (3) for every convex decreasing function Φ on R gives

$$(16) \quad (-\text{Re } g)^* \leq (-\text{Re } h)^* \quad \text{in } D^+.$$

Notice that (15) implies

$$(17) \quad \max_{z \in D} \operatorname{Re} g(z) \leq \max_{z \in D} \operatorname{Re} h(z)$$

and (16) implies

$$(18) \quad \min_{z \in D} \operatorname{Re} h(z) \leq \min_{z \in D} \operatorname{Re} g(z).$$

Moreover, h is univalent and an argument like that used in the proof of Lemma 1 of [8] shows that $h(D)$ is a Steiner symmetric domain.

Hence, Proposition 6 of [8] implies

$$\int_{-\pi}^{\pi} |g(e^{it})|^p dt \leq \int_{-\pi}^{\pi} |h(e^{it})|^p dt, \quad 0 < p \leq 2,$$

which is equivalent to

$$M_p(r, \log f) \leq M_p(r, \log \lambda), \quad 0 < p \leq 2.$$

To prove (12), observe that (11) implies that if f is a Gelfer function then

$$(19) \quad \|\log f\|_{H^p} \leq \|\log \lambda\|_{H^p}, \quad 0 < p \leq 2.$$

Now, if $f \in G$, $w \in D$, and

$$g(z) = f\left(\frac{z+w}{1+\bar{w}z}\right) / f(w),$$

then $g \in G$ and hence

$$(20) \quad \|\log g\|_{H^p} \leq \|\log \lambda\|_{H^p}, \quad 0 < p \leq 2.$$

But notice that

$$\log g = (\log f)_w$$

and then (20) implies that, for $0 < p \leq 2$,

$$\|\log f\|_{H^p} = \sup_{w \in D} \|(\log f)_w\|_{H^p} \leq \|\log \lambda\|_{H^p} = \|\log \lambda\|_{\text{BMOA}_p}.$$

This proves (12), finishing the proof of Theorem 2.

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