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ℕ-COMPACTNESS AND SHAPE

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ABSTRACT. In this paper we prove that two ℕ-compact spaces are homeomorphic if and only if they have the same shape. We also obtain a result concerning shape domination, and finally we give an answer to the problem of components in shape theory.

INTRODUCTION

ℕ-compact spaces were introduced by Mröwka [11] where the general concept of an E-compact space was defined: given a Hausdorff space E, a space X is E-compact if it is homeomorphic to a closed subspace of E^m for some cardinal number m. The ℕ-compact spaces are those that can be embedded as closed subspaces in $\mathbb{N}^m$, where ℕ is the set of natural numbers with the discrete topology.

One of the most important characteristics of E-compact spaces for any given E is that they form an epireflective subcategory of the category of Hausdorff spaces (see [17] for definition).

Other important properties of ℕ-compact spaces are given in [6, 14]. It is clear that an ℕ-compact space X is realcompact and $\operatorname{ind}(X) = 0$, where ind is the small inductive dimension, but the converse is not true as showed by Nyikos [15]. Nyikos proved that Prabir Roy's space Δ [16] is not ℕ-compact (Δ, as showed by Roy, is a metrizable space such that $\operatorname{ind}(\Delta) = 0$ and $\dim(\Delta) = 1$, where dim is the covering dimension). It is known that every realcompact space X with $\dim(X) = 0$ is ℕ-compact. On the other hand, Mröwka [12, 13] proved that the converse is not true. In fact he constructed a space μ and two metrizable subspaces μ_0 and μ_{00} such that they are ℕ-compact and have positive dimension; in particular he pointed out that $\dim(\mu_0) = \dim(\mu_{00}) = 1$.

The main result in this paper assures that two ℕ-compact spaces have the same shape if and only if they are homeomorphic. We also prove that an ℕ-compact space X is shape dominated by an ℕ-compact space Y if and only if X is homeomorphic to a retract of Y. This result allows us to give a

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characterization, involving shape, of those $\mathbb{N}$-compact metrizable spaces with positive dimension (covering dimension).

Let us say that Kozlowski–Segal [7] proved that, as in this paper for $\mathbb{N}$-compactness, two paracompact spaces with null covering dimension have the same shape if and only if they are homeomorphic. On the other hand, we have proved in [10] that two metrizable spaces X, Y such that $\operatorname{ind}(X) = \operatorname{ind}(Y) = 0$ have the same shape if and only if they are homeomorphic. Finally in this paper we give an answer to the problem of components in shape theory. This problem is the following:

Let X, Y be spaces such that $\operatorname{Sh}(X) = \operatorname{Sh}(Y)$ (where Sh is the shape). Is it true that there exists a homeomorphism $K: \square X \to \square Y$ between the corresponding spaces of components such that $\operatorname{Sh}(X_0) = \operatorname{Sh}(K(X_0))$ for every $X_0 \in \square X$? This problem was solved by Borsuk (see [1]) in the compact metric case. Other authors have given answers to this problem but, to the author's knowledge, the more general result concerning this problem can be found in [2].

Information about shape theory can be found in [1, 3, 8, 9].

In this paper map means continuous function. By a clopen set we mean a set that is both open and closed. A clopen ultrafilter is an ultrafilter on the Boolean algebra of clopen subsets of a space.

The proofs of the results in this paper are based on the following characterization of $\mathbb{N}$-compactness first discovered by Herrlich (see [6]).

Theorem 0.1. *A Hausdorff space X with $\operatorname{ind}(X) = 0$ is $\mathbb{N}$-compact if and only if every clopen ultrafilter on X with the countable intersection property is fixed.*

1. On the shape of $\mathbb{N}$-compact spaces

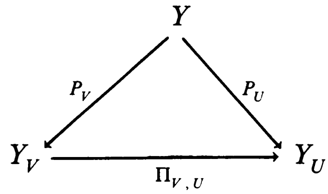
The notion of shape morphism that we use in this paper is that given by Mardešić in [8, p. 267].

The key in order to obtain the announced results is the following.

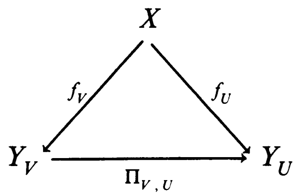
Proposition 1.1. *Let $f: X \to Y$ be a shape morphism from a topological space X to an $\mathbb{N}$-compact space Y. Then there exists a unique map $\Pi(f): X \to Y$ such that $f(\mu) = \mu \cdot \Pi(f)$ for every map $\mu: Y \to Q$ where Q is a discrete space. Moreover if Z is an $\mathbb{N}$-compact space and $g: Y \to Z$ is a shape morphism, then $\Pi(g \cdot f) = \Pi(g) \cdot \Pi(f)$ and Π associates to the identity shape morphism the identity map.*

Proof. Let $D(Y)$ be the family of discrete coverings of Y by nonempty sets. For every pair U, $V \in D(Y)$ we define $U \cap V = \{u \cap v\,;\text{ where } u \in U,\ v \in V,$ and $u \cap v \neq \varnothing\}$. It is clear that $U \cap V \in D(Y)$. With every $U \in D(Y)$ we can associate a discrete space, denoted by Y_U, whose points are the elements of U (the nerve of U). Let $P_U: Y \to Y_U$ be the natural projection. Let us note that if $V \in D(Y)$ is a refinement of U, we can define a map $\Pi_{V,U}: Y_V \to Y_U$ such that for every $\alpha \in V$, $\Pi_{V,U}(\alpha)$ is the unique element of U containing α.

Note that the diagram:



is commutative, then there exist unique maps (because Y_V and Y_U are discrete) $f_U = f(P_U)$; $f_V = f(P_V)$ such that the diagram



is commutative. For every $x \in X$ we have a family $F_x = \{f_U(x),\ U \in D(Y)\}$ of clopen subsets of Y. If $V \in D(Y)$ is a refinement of U then $f_V(x) \subset f_U(x)$, and then $f_U(x) \cap f_T(x) \neq \varnothing$ for every $U, T \in D(Y)$ (because $U \cap T$ is a refinement of both U and T). On the other hand $f_U(x) \cap f_T(x) = f_{U \cap T}(x)$. Obviously the family F_x is a clopen filter on Y. It is clear that for every clopen subset A of Y, it follows that either A belongs to F_x or $Y - A$ belongs to F_x (take $U = \{A, Y - A\}$) and then F_x is a clopen ultrafilter on Y. Let us suppose now that F_x does not have the countable intersection property, then there exists a sequence $U_n \subset D(Y)$ such that

$$\bigcap_{n \in \mathbb{N}} f_{U_n}(x) = \varnothing.$$

Take

$$V_1 = U_1,$$
$$V_n = V_1 \cap V_2 \cap \cdots \cap V_{n-1} \cap U_n \quad \text{for all } n \in \mathbb{N}.$$

It is clear that for every $n \geq 2$, V_n is a refinement of V_{n-1} and U_n. Then we have that $f_{V_n}(x) \subset f_{V_{n-1}}(x) \cap F_{U_n}(x)$ for every $n \geq 2$ and

$$\bigcap_{n=1}^{\infty} f_{V_n}(x) = \varnothing.$$

From this fact, we can suppose without loss of generality that $f_{V_{n+1}}$ is strictly contained in $f_{V_n}(x)$ for all $n \in \mathbb{N}$, and that $f_{V_1}(x)$ is not all Y. Then we can construct a cover

$$T = \{Y - f_{V_1}(x)\} \cup \left(\bigcup_{n=1}^{\infty} \{f_{V_n}(x) - f_{V_{n+1}}(x)\} \right).$$

It is clear that $T \in D(Y)$ and that there exists $n \in \mathbb{N}$ such that $f_T(x) \cap f_{V_n}(x) = \varnothing$ and this is not possible. Consequently we have proved that F_x has the countable intersection property. Now using Theorem 0.1, we have that

$$\bigcap_{U \in D(Y)} f_U(x)$$

is a point in Y, and then we construct the function

$$\Pi(f)\colon X \to Y, \quad \text{as } \Pi(f)(x) = \bigcap_{U \in D(Y)} f_U(x).$$

In order to prove that $\Pi(f)$ is a map, let us suppose that A is a clopen subset of Y containing $\Pi(f)(x)$. Let $U = \{A, Y - A\}$ and let $P_U\colon Y \to Y_U$ be the projection. It is clear that $\Pi(f)(f_U^{-1}(A)) \subset A$ where $f_U = f(P_U)$. Since $\operatorname{ind}(Y) = 0$, it follows that $\Pi(f)$ is continuous. From the construction of $\Pi(f)$ it follows that for every discrete space Q and every map $\mu\colon Y \to Q$ the map $f(\mu)$ is equal to $\mu \cdot \Pi(f)$.

Now let $g\colon Y \to Z$ be a shape morphism with Z $\mathbb{N}$-compact. If Q is discrete space and $\beta\colon Z \to Q$ is a map then $(g \cdot f)(\beta) = \beta \cdot \Pi(g \cdot f)$. On the other hand $(g \cdot f)(\beta) = f(g(\beta)) = f(\beta \cdot \Pi(g)) = \beta \cdot \Pi(g) \cdot \Pi(f)$; then $\Pi(g) \cdot \Pi(f) = \Pi(g \cdot f)$ because every pair of points in Z are separated by a clopen subset. Obviously Π associates to the identity shape morphism the identity map and proof is finished.

As consequences of the last proposition we have:

Corollary 1.2. *Let X, Y be two $\mathbb{N}$-compact spaces, then* $\operatorname{Sh}(X) \geq \operatorname{Sh}(Y)$ *if and only if Y is homeomorphic to a retract of X.*

Proof. Let us suppose that $\operatorname{Sh}(X) \geq \operatorname{Sh}(Y)$; then there exist shape morphisms $f\colon X \to Y$, $g\colon Y \to X$ such that $f \cdot g = I_Y$ (identity shape morphism). It follows that $\Pi(f) \cdot \Pi(g) = 1_Y$ and then $\Pi(g)$ is a homeomorphism onto $\Pi g(Y)$. On the other hand the map $\Pi(g) \cdot \Pi(f)\colon X \to \Pi(g)(Y)$ is a retraction. The converse is obvious.

Corollary 1.3. *Let X, Y be two $\mathbb{N}$-compact spaces. Then* $\operatorname{Sh}(X) = \operatorname{Sh}(Y)$ *if and only if X and Y are homeomorphic.*

Using Corollary 1.2 we give the following characterization.

Corollary 1.4. *Let X be a metrizable $\mathbb{N}$-compact space, then* $\dim X > 0$ *if and only if there exists a closed subset A of X such that the inclusion map $i\colon A \to X$ is not left invertible in the shape category.*

Proof. Let us suppose that $\dim X > 0$; then there exists a closed subset A of X and a map $f\colon A \to S^0$ ($S^0 = \{0, 1\}$) which is not extendable to X (see [4] for example) then A is not a retract of X. If $i\colon A \to X$ were left invertible in the shape category then, using Corollary 1.2, A is a retract of X. On the other hand, if $\dim X = 0$, and since X is metrizable (see [5] for example) then every closed subspace A of X is a retract of X and then $i: A \to X$ is left invertible. □

Using the same arguments as in the construction of $\Pi(f)$ in Proposition 1.1 and applying Lemma 1 in [3] we have:

Proposition 1.5. *Suppose $f: X \to Y$ is a shape morphism, Y paracompact, $\square Y$ $\mathbb{N}$-compact, and the projection $P_Y: Y \to \square Y$ closed. Then there exists a unique map $K_f: \square X \to \square Y$ such that for every component X_0 of X there exists a unique shape morphism $f_0: X \to K_f(X_0)$ such that $\mathrm{S}(j) \cdot f_0 = f \cdot \mathrm{S}(i)$, where $i: X_0 \to X$ and $j: K_f(X_0) \to Y$ are the inclusions and $\mathrm{S}(h)$ denotes the shape morphism induced by the map h.*

Corollary 1.5. *Let X, Y be paracompact spaces such that $\square X, \square Y$ are $\mathbb{N}$-compact spaces and suppose that the projections $P_X: X \to \square X$, $P_Y: Y \to \square Y$ are closed. If $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ then there exists a homeomorphism $K: \square X \to \square Y$ such that $\mathrm{Sh}(X_0) = \mathrm{Sh}(K(X_0))$ for every component $X_0 \in \square X$.*

In order to end we ask the following:

Problem 6. What relation exists between the shape morphisms f and $\mathrm{S}(\Pi(f))$? When $f = \mathrm{S}(\Pi(f))$?

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