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A FUNCTION WHICH IS ARC-ANALYTIC BUT NOT CONTINUOUS

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ABSTRACT. We construct an arc-analytic function (i.e., a function analytic on every real analytic arc) which is not continuous, as well as a continuous arc-analytic function which is not subanalytic.

A function $f(x)$ of n real variables $x = (x_1, \dots x_n)$ is called *arc-analytic* if $f(x(t))$ is analytic for every real analytic arc $x = x(t)$. In this note, we construct:

(1) an arc-analytic function $f(x_1, x_2)$ which is not continuous; and

(2) a continuous arc-analytic function $f(x_1, x_2)$ whose graph is not subanalytic.

These examples answer questions raised in [1]. According to [1, Theorem 1.4], $f(x_1, \dots, x_n)$ is arc-analytic and has subanalytic graph if and only if it can be transformed to an analytic function by composition with finite sequences of local blowings-up in x with smooth centres. Our example (2) shows that "subanalytic" is not a superfluous hypothesis in this result. Every arc-analytic function with subanalytic graph is continuous.

The idea of our construction is the following: Let $Z_0 = \mathbb{R}^2$, and fix a sequence of blowings-up $\pi_i \colon Z_i \to Z_{i-1}$ with one-point centres. For simplicity, assume that these centres are determined by the germ at the origin of the curve $\gamma_0(t) = (t, 0) \in \mathbb{R}^2$. This means that $\pi_1 \colon Z_1 \to \mathbb{R}^2$ is the blowing-up of the origin and, if γ_i denotes the strict transform of γ_{i-1} by π_i, then $\pi_{i+1} \colon Z_{i+1} \to Z_i$ is the blowing-up with centre $\gamma_i(0)$. Take a sequence of arc-analytic funitons H_n such that each H_n becomes analytic after exactly n blowings-up; i.e., $H_n \circ \pi_1 \circ \cdots \circ \pi_i$ is analytic if and only if $i \geq n$. Then $H = \sum_{n=1}^{\infty} a_n H_n$, where $\{a_n\}$ is a rapidly decreasing sequence of positive numbers, is a candidate for a function which is arc-analytic but not subanalytic.

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Consider, for example,

$$H_n(x, y) = \begin{cases} xh_n(x, y), & (x, y) \neq 0, \\ 0, & (x, y) = 0, \end{cases}$$

where

$$h_n(x, y) = x^{2n}/(x^{2n} + y^2), \qquad (x, y) \neq 0,$$

$n = 1, 2, \dots$. It is easy to check that h_n is analytic (i.e., extends to be analytic) on every analytic arc and, moreover, that h_n becomes analytic after exactly n blowings-up. However, on the line $\{x = c\}$, h_n restricts to $\phi_n(y) = c^{2n}/(c^{2n} + y^2)$. If $0 < |c| < 1$, then the complex roots of the denominators tend to 0 as $n \to \infty$, so that $\sum_{n=1}^{\infty} a_n \phi_n(y)$ cannot converge to an analytic function. The problem here is that the complex zero sets of the denominators $f_n(x, y) = x^{2n} + y^2$ become closer and closer to the real plane as $n \to \infty$.

We will modify the f_n in such a way that their complex zeros do not change much as n increases; in particular, the distances from their complex zero sets to a nonzero point $(x_0, y_0) \in \mathbb{R}^2$ will be bounded away from zero.

We first define a sequence of polynomials $g_n(x, y)$ inductively by the formulas

$$g_1 = f_1,$$
$$g_n = \delta_n f_n g_1^{n-2} g_2^{n-3} \cdots g_{n-2}^{1} + g_{n-1}^{2}, \qquad n = 2, 3, \dots,$$

where the $\delta_n > 0$ is specified below. Then we set

$$h_n(x, y) = x^{2^n}/g_n(x, y), \qquad n = 1, 2 \dots.$$

We describe how the f_n and g_n transorm by the blowings-up π_i: To begin with, Z_1 is covered by 2 coordinate charts U_1 and U_1', each isomorphic to $\mathbb{R}^2$, in which π_1 is given by the following formulas:

$$\pi_{11} = \pi_1|U_1 \colon (x_1, y_1) \mapsto (x_1, x_1 y_1),$$
$$\pi_{12} = \pi_1|U_1' \colon (x_1', y_1') \mapsto (x_1' y_1', y_1').$$

Therefore

$$f_n \circ \pi_{11} = x_1^2 f_{n-1}, \qquad n > 1,$$
$$f_n \circ \pi_{12} \sim y_1'^2$$

(where $g \sim h$ means that g equals h times a factor that vanishes nowhere). Thus the strict transform f_{1n} of f_n by π_1 vanishes only in U_1. (The strict transform, which is defined locally up to an invertible factor, is given in U_1 by $f_n \circ \pi_{11}/x_1^2$ and in U_1' by $f_n \circ \pi_{12}/y_1'^2$.) Proceeding inductively, we see that for each $i = 1, 2, \dots$, Z_i has one chart $U_i = \mathbb{R}^2$ with coordinates (x_i, y_i) such that:

(1) $\pi_i(U_i) \subset U_{i-1}$ and $\pi_i|U_i$ is given by

$$(x_{i-1}, y_{i-1}) = (x_i, x_i y_i), \qquad i = 2, 3, \dots.$$

(2) The strict transform f_{in} of f_n by $\pi_1 \circ \pi_2 \circ \cdots \circ \pi_i$ vanishes only in U_i, and

$$f_{in}|U_i = f_{n-i}, \qquad n > i,$$
$$f_{in}|U_i \sim 1, \qquad n \le i.$$

Lemma. *The functions g_n vanish only at the origin. For each $n = 2, 3, \ldots$, the function g_n has multiplicity 2^{n-1} at 0. (g_1 has multiplicity 2 at 0.) For each n, the strict transform g_{in} of g_n by $\pi_1 \circ \cdots \circ \pi_i$ vanishes only at the origin of U_i, and on U_i,*

$$g_{in} \sim 1, \qquad n \le i,$$

$g_{i,i+1}$ has a nondegenerate critical point at 0, and if $n > i+1$, then

$$g_{in} = \delta_n f_{in} g_{i1}^{n-2} g_{i2}^{n-3} \cdots g_{i,n-2}^{1} + g_{i,n-1}^{2},$$

so that g_{in} has multiplicity 2^{n-i-1} at 0.

Proof. By induction on n. □

From the lemma, on U_1 we get

$$h_n \circ \pi_1 = (x^{2^n}/g_n) \circ \pi_1 \sim x_1^{2^{n-1}}/g_{1n},$$

and, in general, on U_i we obtain

$$h_n \circ \pi_1 \circ \cdots \circ \pi_i \sim x_i^{2^{n-i}}/g_{in}, \qquad n > i.$$

Proposition. *If the δ_n are sufficiently small and $\{a_n\}$ is a rapidly decreasing sequence of positive numbers, then $h = \sum_{n=1}^{\infty} a_n h_n$ is analytic on every analytic arc, and*

$$H(x, y) = \begin{cases} xh(x, y), & (x, y) \ne 0, \\ 0, & (x, y) = 0, \end{cases}$$

defines an arc-analytic function.

Proof. We will show that the δ_n can be chosen so that, if $\{a_n\}$ is rapidly decreasing, then $h \circ \pi_1 \circ \cdots \circ \pi_i$ is analytic on $Z_i - \{\gamma_i(0)\}$, $i = 1, 2, \ldots$. This guarantees that h is analytic on any analytic arc except perhaps γ_0. On the other hand, $g_n(x, 0) = x^{2^n} g_n(1, 0)$, where $g_n(1, 0) > 0$, $n = 1, 2, \ldots$, so that $h(x, 0)$ is constant on γ_0. Therefore H is an arc-analytic function.

Let X_n denote the complex zero set of g_n, $n = 1, 2, \ldots$. If we choose δ_2 small enough, we can make X_2 close to X_1; then choosing δ_3 small enough, we can make X_3 chose to X_2, etc. More precisely: For each $k = 1, 2, \ldots$, put

$$V_k = \{(x, y) \in \mathbb{R}^2 : 1/k^2 < x^2 + y^2 < 1\}$$

and

$$c_k = \tfrac{1}{2} d(V_k, X_1), \qquad k = 1, 2, \ldots$$

(d denotes the Euclidean distance). Choose the δ_n successively, small enough so that $d(V_k, X_n) \ge c_k$ for all $k = 1, \ldots, n$, $n = 1, 2, \ldots$. With such a choice of the δ_n (and with $\{a_n\}$ rapidly decreasing), h will be analytic in each annulus V_k; hence analytic in $0 < x^2 + y^2 < 1$.

In fact, we can choose the δ_n so that, also for each $i = 1, 2, \dots$, $h \circ \pi_1 \circ \cdots \circ \pi_i$ is analytic in a punctured neighborhood of the point $\gamma_i(0)$ (i.e., of the origin in the chart U_i): Let X_{in} denote the complex zero set of $g_{in}|U_i$ for each $n > i$, $i = 1, 2, \dots$. Put

$$V_{ik} = \{(x_i, y_i) \in U_i : 1/k^2 < x_i^2 + y_i^2 < 1\}$$

and

$$c_{ik} = \tfrac{1}{2} d(V_{ik}, X_{i,i+1}).$$

It suffices to choose the δ_n so that also $d(V_{ik}, V_{in}) \geq c_{ik}$ for all $k = 1, \dots, n$ and all $i = 1, \dots, n$.

On the other hand, consider any coordinate chart $\widetilde{U}_i = \mathbb{R}^2$ of Z_i, $i = 0, 1, \dots$, and any compact subset K of $\widetilde{U}_i$ that does not include $\gamma_i(0)$. Then the strict transform $g_{i,i+1}$ of g_{i+1} by $\pi_i \circ \cdots \circ \pi_i$ vanishes nowhere on a complex neighborhood L of K in $\mathbb{C}^2$. It follows that we can choose the δ_n also small enough that, for all such $\widetilde{U}_i$ and K, $i = 0, 1, \dots$, the strict transorms g_{in} of the g_n by $\pi_1 \circ \cdots \circ \pi_i$ vanish nowhere on a fixed complex neighborhood of K, for all $n > i$; therefore (if $\{a_n\}$ is rapidly decreasing) $h \circ \pi_1 \circ \cdots \circ \pi_i$ converges to an analytic function in a neighborhood of K.

To sum up, the δ_n can be chosen so that, if $\{a_n\}$ is rapidly decreasing, then $h \circ \pi_1 \circ \cdots \circ \pi_i$ is analytic on $Z_i - \{\gamma_i(0)\}$, $i = 0, 1, 2, \dots$. □

Example (2). Take H as in the proposition above. Clearly, H is continuous. It is easy to see that each $h_n \circ \pi_1 \circ \cdots \circ \pi_i$ is analytic precisely when $i \geq n$, and the directional derivative of $\sum_{n>i} a_n h_n \circ \pi_1 \circ \cdots \circ \pi_i$ at 0 in the direction (x, y) equals $a_{i+1} x^2 / Q_i(x, y)$, where Q_i is a nondegenerate quadratic form (Q_i is the initial form of $g_{i,i+1}$). Therefore, $H \circ \pi_1 \circ \cdots \circ \pi_i$ is never analytic. It follows from [1, Theorem 1.4] that the graph of H is not subanalytic.

Example (1). Take h as in the proposition. Clearly,

$$g_n(x, y) > g_n(x, 0) = x^{2^n} g_n(1, 0),$$

where $g_n(1, 0) > 0$, for all x, all $y \neq 0$, and $n = 1, 2, \dots$; therefore,

$$h_n(x, y) < h_n(x, 0) = 1/g_n(1, 0) = h_n(1, 0),$$

for all $x \neq 0$, $y \neq 0$, and $n = 1, 2, \dots$. Put

$$f(x, y) = \begin{cases} y e^{2/(x^2 + h(1,0) - h(x,y))}, & (x, y) \neq 0, \\ 0, & (x, y) = 0. \end{cases}$$

We claim that $f(x, y)$ is an arc-analytic function but that, if $\{a_n\}$ is sufficiently rapidly decreasing, then

$$\lim_{x \to 0} \frac{h(1, 0) - h(x, e^{-1/x^2})}{x^2} = 0,$$

so that $\lim_{x\to 0} f(x, e^{-1/x^2}) = +\infty$ and f is not continuous at 0.

Indeed, for every $n = 1, 2, \dots$,

$$0 \le h_n(1, 0) - h_n(x, y) \le \frac{y^2}{x^{2^n} g_n(1, 0)^2} \cdot \frac{g_n(x, y) - g_n(x, 0)}{y^2}.$$

Since the second fraction on the right is a polynomial in (x, y), $h_n(1, 0) - h_n(x, e^{-1/x^2})$ is flat at 0. Therefore, if $\{a_n\}$ is rapidly decreasing, $h(1, 0) - h(x, e^{-1/x^2})$ is flat at 0, so that $\lim_{x\to 0}(h(1, 0) - h(x, e^{-1/x^2}))/x^2 = 0$.

To show that $f(x, y)$ is arc-analytic, it suffices to prove

$$h(1, 0) - \lim_{t\to 0} h(x(t), y(t)) > 0,$$

for any analytic arc $\sigma_0(t) = (x(t), y(t))$ such that $\sigma_0(0) = 0$ and $y(t) \not\equiv 0$. Let σ_i denote the strict transform of σ_{i-1} by $\pi_i: Z_i \to Z_{i-1}$, $i = 1, 2, \dots$. (In particular $\sigma_{i-1} = \pi_i \circ \sigma_i$.) There is a positive integer r such that $\sigma_i(0) = \gamma_i(0)$, $0 \le i < r$, and $\sigma_r(0) \ne \gamma_r(0)$. If $n > i$, then $(h_n \circ \pi_1 \circ \cdots \circ \pi_i)(c) = 0$ for all $c \in \pi_i^{-1}(\gamma_{i-1}(0))$, $c \ne \gamma_i(0)$ (since $h_n \circ \pi_1 \circ \cdots \circ \pi_i \sim x_i^{2^{n-i}}/g_{in}$ on U_i). Therefore, $h_n(x(0), y(0)) = 0$ for all $n > r$, and

$$h(1, 0) - h(x(0), y(0)) \ge \sum_{n=r+1}^{\infty} \frac{a_n}{g_n(1, 0)} > 0.$$

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