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ON A BASIS FOR $H_2(\overline{M}_g)$

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ABSTRACT. In the moduli space $\overline{\mathbf{M}}_{\mathbf{g}}$ of stable curves (Riemann surfaces with nodes) we construct a basis for the second homology group, which is dual to the standard basis for the second cohomology group. The elements of our basis are algebraic curves.

0. INTRODUCTION

Let $\mathbf{M}_{\mathbf{g}}$ denote the moduli space of nonsingular complex curves (compact Riemann surfaces) of genus g and $\overline{\mathbf{M}}_{\mathbf{g}}$ its stable curves compactification. The goal of this note is to provide a set of $2+[g/2]$ complete curves (compact analytic subspaces of complex dimension 1), $E, E_0, \ldots, E_{[g/2]}$ that pair diagonally with the basic divisor classes $\lambda, \delta_0, \ldots, \delta_{[g/2]}$ (see [HM]). A fundamental result of Harer (see [Wo]) implies at once that these curves afford a basis for $H_2(\overline{\mathbf{M}}_{\mathbf{g}})$.

From now on, we restrict ourselves to $g > 3$, because our method to construct a curve E_0 dual to D_0 does not work for $g = 3$ (but see Remark 1). As for the curves E_i $i \geq 1$, they are, of course, those introduced by Mumford and Harris [HM].

Our work is closely related to the article of Wolpert [Wo]. As in that paper we employ the Teichmüller coordinates for $\overline{\mathbf{M}}_{\mathbf{g}}$ defined by Bers in [Be]. Apart from the fact that our intersection table is diagonal, our method is different in that we avoid homotopy theory. Instead we determine the restriction of these divisors to our chosen curves. In this way, we remain within the framework of analytic (algebraic) geometry.

1. PRELIMINARIES

1.1. $\overline{\mathbf{M}}_{\mathbf{g}}$ is the moduli space of stable curves (Riemann surfaces with nodes [Be]). It is a compactification of $\mathbf{M}_{\mathbf{g}}$ whose compactification locus, $\overline{\mathbf{M}}_{\mathbf{g}} - \mathbf{M}_{\mathbf{g}}$, is the union of $1+[g/2]$ divisors $D_0, \ldots, D_{[g/2]}$. These can be readily described

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in terms of the generic point, i.e. by specifying a Zariski, and hence dense, open set.

The generic point of the divisor D_i, $i \geq 1$, represents the stable curve obtained by glueing together arbitrary nonsingular curves S_1, S_2 of genus i and $g-i$ respectively along points $P \in S_1$, $Q \in S_2$; we shall denote it by $S_1 \bigvee_{P=Q} S_2$.

As for D_0, its generic point represents a nonsingular curve of genus $g-1$ where two points become identified. Intersection of divisors only occurs at points representing stable curves with two or more nodes, i.e. with two or more pairs identified.

$\mathbf{M_g}$ and $\overline{\mathbf{M}}_\mathbf{g}$ are virtual manifolds (V-manifold), which in essence, means that each point has a neighborhood $\overline{V}$ of the form V/G, where V is an open set in $\mathbf{C^n}$, G is a finite group of biholomorphic transformations and there is a continuous map $\varphi: V \to \overline{V}$, which is G-equivariant and induces a homeomorphism from V/G onto $\overline{V}$. The triple (V, G, φ) is called a local uniformizing system (l.u.s.) (for the precise definition of V-manifold, see [Sa]).

Bers [Be, §3, pp. 46–47] has given a very explicit collection of l.u.s. for $\overline{\mathbf{M}}_\mathbf{g}$, which we now quote.

Let S be a stable curve and [S] the corresponding point in $\overline{\mathbf{M}}_\mathbf{g}$. A l.u.s. (V, G, φ) for [S] is constructed as follows:

(i) Let S have r components $S_1, \dots, S_r$ and k nodes. Choose Fuchsian groups $G_1, \dots, G_r$ acting on discs $U_1, \dots, U_r$ with disjoint closures in $\mathbf{P}^1$ such that (a) G_j has n_j nonconjugate maximal elliptic subgroups, each of the same fixed order ≥ 3 with $n_j = \#$ punctures of S_j-{nodes}. (b) $U_j^*/G_j = U_j/G_j$-{image of elliptic fixed points} is isomorphic to $S_j^* = S_j$-{nodes}.

(ii) $G_1, \dots, G_r$ generate a Kleinian group G, which is their free product, such that G has precisely one invariant component U^0.

(iii) For each node P_i, we can assign two nonconjugate maximal elliptic subgroups Γ_i', Γ_i'' corresponding to the two punctures P_i', P_i'' determined by P_i. If $P_i' \in S_j$ and $P_i'' \in S_l$ it is assumed that $\Gamma_i' \subset G_j$ and $\Gamma_i'' \subset G_l$.

(iv) Call two elliptic fixed points not in U^0 *related* if they are fixed under elliptic subgroups conjugate to Γ_i' and Γ_i'' for some i. Then the union of the U_j/G_j, with the images of any pair of related elliptic fixed points identified, is isomorphic to S.

(v) Let g_{i,s_i} be the unique loxodromic transformation that conjugates Γ_i' into Γ_i'', has multiplier s_i, $|s_i| > 0$ and small, and has fixed points in U_j and U_l (j and l as before).

(vi) Now, the group generated by G and the transformations g_{i,s_i}, $i = 1, \dots, k$ with $s_i \neq 0$ is a Kleinian group. Let us call it $G_{\mathbf{0},\mathbf{s}}$ where $\mathbf{s}(s_1, \dots, s_k)$.

(vii) Let $\mathbf{s}$ be as before, and let w be a quasiconformal automorphism of $\mathbf{C}$ such that w leaves 0, 1, ∞ fixed, $w|_{U^0}$ is conformal, and $w \cdot G_{\mathbf{0},\mathbf{s}} \cdot w^{-1}$ is a Kleinian group. Then $w|_{U_j}$, $j = 1, \dots, r$ defines an element τ_j of the

Teichmüller space $\mathbf{T}(G_j)$; set $\boldsymbol{\tau} = (\tau_1, \dots, \tau_r)$. If $s_i \neq 0$, set $\varepsilon_i = a_i - \hat{a}_i$, where

$$a_i = \text{repelling fixed point of } w \cdot g_{i,s_i} \cdot w^{-1},$$

$$\hat{a}_i = \text{fixed point of } w \cdot \Gamma'_i \cdot w^{-1} \text{ in } U_j.$$

If $s_i = 0$, set $\varepsilon_i = 0$. Also set $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_k)$. The point $(\boldsymbol{\tau}, \boldsymbol{\varepsilon}) \in \mathbf{C}^{3\mathbf{g}-3}$ determines the group $w \cdot G_{\mathbf{0};\mathbf{s}} \cdot w^{-1}$ completely; we denote this group by $G_{\boldsymbol{\tau},\boldsymbol{\varepsilon}}$.

(viii) Then the following triple (V, G, φ) is a l.u.s. for a neighborhood of [S].

- V is a small neighborhood of the origin in $\mathbf{C}^{3\mathbf{g}-3}$. (We observe that the origin corresponds to [S].)
- $G = \mathrm{Aut}(S)$, the group of conformal automorphisms of S [Be, XII, p. 52].
- The map $\varphi: V \to \overline{\mathbf{M}}_{\mathbf{g}}$ is defined as follows [Be, I, p. 47]. The Kleinian group $G_{\boldsymbol{\tau},\boldsymbol{\varepsilon}}$ defines for us a Riemann surface; if some $\varepsilon_i = 0$ we agree to identity the corresponding pairs of related fixed points. Thus, $\hat{a}_i$, $w \cdot g_{i,s_i} \cdot w^{-1}(\hat{a}_i)$ (and the points related to these two) become identified to create a node. In particular the stable curve $S_{\boldsymbol{\tau},\boldsymbol{\varepsilon}}$ so constructed, has as many nodes as zeros are in $\varepsilon_1, \dots, \varepsilon_k$. We set

$$\varphi(\boldsymbol{\tau}, \boldsymbol{\varepsilon}) = [S_{\boldsymbol{\tau},\boldsymbol{\varepsilon}}].$$

(ix) For our purposes it is useful to require that $\mathbf{T}(G_r)$ be described as follows:

With the notation as above, let us assume that S_r^* has m punctures and genus $g' > 1$ (this will always be the case in what follows). Let $\mathbf{T}_{p,n}$ denote the Teichmüller space of the surface of genus p with n punctures; and let $\mathbf{V}_p$ be the Teichmüller curve. Recall that $\mathbf{V}_p$ is a fibre space over $\mathbf{T}_p$, $\pi: \mathbf{V}_p \to \mathbf{T}_p$, such that for each $t \in T_p$, the fibre $\pi^{-1}(t) = S_t$ is the Riemann surface represented by t.

Then, it is well known that $T(G_r)$ is isomorphic to $\mathbf{T}_{g',m}$ (Bers–Greenberg isomorphism theorem, see [Na, 2.2.8]), and that coordinates for $\mathbf{T}_{g',m}$ with $g' > 1$, can be locally written as

$$\tau_r = (t_r, z_1, \dots, z_m),$$

where $t_r \in \mathbf{T}_g$, parametrizes the same Riemann surface as $\tau_r \in \mathbf{T}_{g',m}$ (disregarding the punctures) and $z_1, \dots, z_m \in S_t$ parametrize the punctures.

1.2. On a V-manifold such as $\overline{\mathbf{M}}_{\mathbf{g}}$, a divisor (complex analytic subspace of codimension 1) is given on each l.u.s. as the zero locus of a holomorphic function. For instance, our divisors D_i, $i = 0, \dots, [g/2]$ generically have local equations $\varepsilon_d = 0$, for suitable d.

In the classical case, given a complex manifold X and a divisor D in it; one associates with D, the line bundle $L(D)$ defined by having as transition functions, quotients of local equations for D.

In the V-manifold case the analogous procedure for the l.u.s. defines what is called a V-bundle [Ba1, §2; Ba2, §3, p. 408].

Following Mumford (see [HM]) we denote by δ_i, $i = 0, \ldots, [g/2]$ the V-bundles associated to the divisors D_i (with the V-manifold structure for $\overline{\mathbf{M}}_{\mathbf{g}}$ and local equations for D_i as described above). The V-line bundle λ, the Hodge bundle [HM], is the gth exterior power of the rank g V-bundle H over $\overline{\mathbf{M}}_{\mathbf{g}}$, whose fibre at the point representing a stable curve S is the vector space of *regular* 1*-forms* on S. These are 1-forms holomorphic on S-{nodes} and having poles with opposite residues at the paired punctures [Be, §1]. We remark in passing that statement VII in [Be], provides for the required local trivializations of H to give a rigorous definition of H in terms of Bers's own coordinates.

V-manifolds are almost as nice as smooth manifolds in the sense that one readily makes sense on a V-manifold Y of concepts such as de Rham cohomology groups, integration of forms, sections, metrics, and curvature (or Chern class) of a line bundle L, etc. In particular, we can define *intersection numbers* by

$$L \cdot E \int_E c_1(L),$$

where E is a second homology class and $c_1(L) \in H^2_{DR}(Y)$ is the Chern class of L.

Good references for all this are [GH, Chapter 1, §1] in the manifold case, and [Sa, Ba1, §2; Ba2, §2.3] for the corresponding facts in the V-case.

We must say, however, that in the cases occurring in this article, E is a Riemann surface and hence $L \cdot E =$ degree of the restriction $L_{|E}$ (see [GH, p. 144]). The reader, who so wants, may take this as a definition.

2. Construction of the dual curves

Next we construct our homology classes, which will in fact be complete curves. But first, we recall that there is another well-known compactification of $\mathbf{M}_{\mathbf{g}}$, the Satake compactification, which we denote by $\mathbf{M}^*_{\mathbf{g}}$, which is obtained by first embedding $\mathbf{M}_{\mathbf{g}}$ in *Siegel space* via the period map, and then embedding Siegel space in projective space $\mathbf{P}^{\mathbf{n}}$ by means of *Siegel modular forms*. Then $\mathbf{M}^*_{\mathbf{g}}$ is the Zariski closure of the biholomorphic image of $\mathbf{M}_{\mathbf{g}}$ in $\mathbf{P}^{\mathbf{n}}$ so obtained (see [Fr]).

We do not need a detailed description of $\mathbf{M}^*_{\mathbf{g}}$; we just remind the reader of the definition of some of the concepts involved.

Denote by $\mathbf{H}_{\mathbf{g}}$ the set of $g \times g$ complex matrices Ω that are symmetric and have positive definite imaginary part. The symplectic group $S_p(g, \mathbf{Z})$ acts properly discontinuously on $\mathbf{H}_{\mathbf{g}}$ by $M \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}$, where

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in S_p(g, \mathbf{Z}).$$

The quotient space $\mathbf{H}_{\mathbf{g}}/S_p(g, \mathbf{Z})$ is the Siegel space.

Let $t \in \mathbf{M_g}$; in a l.u.s. one can make a continuous choice of a canonical basis for the Riemann surface S_t represented by t, $\{A_1(t), \ldots, A_g(t); B_1(t), \ldots, B_g(t)\}$. The matrix $\Omega_t = (\int_{B_j(t)} V_i(t))$ where $v_1(t), \ldots, v_g(t)$ is the dual basis for the holomorphic 1-forms, is called the period matrix. The rule that associates to each $t \in \mathbf{M_g}$, the matrix $\Omega_t \in \mathbf{H_g}$, is a (multivalued) holomorphic map (see [Na]) called the period map.

Let us suppose that a different choice is made for the canonical homology basis of the same Riemann surface S_t, let $v_1'(t), \ldots, v_g'(t)$ be the corresponding dual basis, and Ω_t' the corresponding period matrix. Then it is well known (see [FK]) that the two homology bases are related by the matrix $M \in S_p(g, \mathbf{Z})$ and that

$$(1) \qquad \left\{ \begin{matrix} v_1'(t) \\ \vdots \\ v_g'(t) \end{matrix} \right\} = (C\Omega_t + D)^{-1} \left\{ \begin{matrix} v_1(t) \\ \vdots \\ v_g(t) \end{matrix} \right\},$$

$$(2) \qquad \Omega_t' = M \cdot \Omega_t .$$

Thus, the induced map from $\mathbf{M_g}$ to $\mathbf{H_g}/S_p(g, \mathbf{Z})$ is single valued; and by Torelli's theorem it is also injective.

We also recall the definition of Siegel modular forms; these are holomorphic functions F on $\mathbf{H_g}$ satisfying

$$(3) \qquad F(M \cdot \Omega) = \det(C\Omega + D)^k F(\Omega).$$

The fundamental result is that by taking a basis $F_0, \ldots, F_n$ of the vector space of modular forms of suitable (fixed) weight k, the map $\Omega \to (F_0(\Omega), \ldots, F_n(\Omega))$ embeds Siegel space in projective space $\mathbf{P^n}$ (see [Fr; BPV, p. 108]).

Relevant for us is the fact (see [H]) that for $g \geq 3$ $\mathbf{M_g^*} - \mathbf{M_g}$ has codimension 2, hence a curve can be obtained inside the moduli space $\mathbf{M_g}$ as the generic intersection of $3g - 4 = \dim \mathbf{M_g} - 1$ hypersurfaces in $\mathbf{M_g^*}$.

(I) We take the first curve of our basis to be a fixed curve inside $\mathbf{M_g}$. Let us call it E.

(II) Fix a Riemann surface S of genus $g - i$ without automorphisms, a Riemann surface S^i of genus i, and a point P on it. Then the curve E_i, $i = 1, \ldots, [g/2]$ is the image in D_i of the map

$$\Phi_i : S \to \overline{\mathbf{M}}_\mathbf{g}$$
$$Q \to \left[S^i \bigvee_{P=Q} S \right].$$

Thus the varying position on S of the node describes the point of the curve E_i.

(III) To construct our last curve E_0, we take a Riemann surface of genus $g-1$ possessing an automorphism $\sigma : S \to S$ without fixed points. For instance if

X_1 is a Riemann surface of genus 2 uniformized by a freely acting fuchsian group $\Gamma = \langle a_1, a_2; b_1, b_2\rangle$, then the normal closure Γ_n of $\langle a_1^n, a_2; b_1, b_2\rangle$, uniformizes a Riemann surface $X_n = U/\Gamma_n$, which is naturally a smooth cyclic cover with Galois group $\Gamma/\Gamma_n = \langle a_1\Gamma_n = \sigma\rangle \approx \mathbf{Z}/n\mathbf{Z}$. By Riemann-Hurwitz its genus is $n+1$.

Then the curve E_0 is taken to be the image in D_0 of the map

$$\Phi_0 : S \to \overline{\mathbf{M}}_{\mathbf{g}},$$

which sends $Q \to [S_Q]$, where S_Q stands for the stable curve obtained by identifying Q and $\sigma(Q)$.

Note 1. If $g = 3$ the Riemann surface S required for constructing E_0 would have genus 2, thereby not possessing automorphisms without fixed points and this definition does not work (but see Remark 1).

Note 2. For the cases E_1 in genus 3 and E_3 in genus 4 the required Riemann surface has genus 2, thereby always admitting at least the hyperelliptic involution. In this case we agree to allow only this automorphism.

The maps Φ_i, $i \geq 0$ have very simple local expressions in terms of the coordinates described in §1 (ix).

($i \geq 1$). If z is a local coordinate near $Q \in S$.

$$\Phi_i(z) = (\tau_1^0, t_2^0; 0),$$

where τ_1^0 (resp. t_2^0) is the coordinate for the punctured surface $S^i - \{P\}$ (resp. for the surface S). Here S^i, S, the point P and $\varepsilon = 0$ are fixed; the only thing that is varying is $z(Q)$.

($i = 0$). Similarly

$$\Phi_0(z) = (t_1^0, z \cdot \sigma^{-1}, z; 0).$$

Here t_1^0 is a coordinate for S, which remains fixed, z is a local coordinate near $Q \in S$, and hence $z \cdot \sigma^{-1}$ is a local coordinate near $\sigma(Q)$.

We observe that the maps Φ_i are holomorphic and nonconstant. Thus, their images are complete curves (Proper Mapping Theorem).

3. Computing intersections

3.1. We first identify the restriction of the fundamental classes λ, δ_i, $i = 0, \dots, [g/2]$ to our curves E_i, or rather their pull backs $\Phi_i^*(\lambda)$, $\Phi_i^*(\delta_i)$.

Theorem 1. *Let S be the Riemann surface occurring in the cycle E_i for $i = 0, \dots, [g/2]$. Then*

(i) $\Phi_i^*\delta_i = T_S$, *the tangent bundle over* S.

(ii) $\Phi_i^*\lambda$ *is the trivial bundle over* S.

Proof. (i) Let (U_α, z_α) and (U_β, z_β) be two overlapping charts in S. Their images via Φ_i lie respectively in neighborhoods V_α, V_β.

Let $\varepsilon_\alpha = 0$ (resp. $\varepsilon_\beta = 0$) be the defining equation for D_i in V_α (resp. in V_β). According to the definition of δ_i given in §1.2 the transition functions for $\Phi_i^*\delta_i$ are

$$\Omega_{\alpha\beta}(Q) = \varepsilon_\beta/\varepsilon_\alpha|_{\Phi_i(Q)}; \qquad Q \in U_\alpha \cap U_\beta .$$

To understand this quotient, we must go back to the very definition of Bers's coordinates.

Let $(\tau_1, t_2, z_\alpha(Q); \varepsilon_\alpha)$, $(\tau_1, t_2, z_\beta(Q); \varepsilon_\beta)$, with $\varepsilon_\alpha \neq 0$, $\varepsilon_\beta \neq 0$, be the α and β-coordinates of a point near $\Phi_i(Q)$ respectively. By going through the identifications made in §1.1 (ix) we see that Q corresponds to the $\hat{a}_i$ of §1 (vii) and, that if $Q' \in U_\alpha$ corresponds to the a_i of §1 (vii) then the equation $z_\alpha(Q') - z_\alpha(Q) = 0$ is locally defining for D_i. Thus the line bundle determined by this collection of functions over S is equivalent to the line bundle determined by the ε_α's.

We can now write

$$\Omega_{\alpha\beta}(z_\alpha(Q)) = \left.\frac{z_\beta(Q') - z_\beta(Q)}{z_\alpha(Q') - z_\alpha(Q)}\right|_{\Phi_i(Q)} = \lim_{Q' \to Q} \frac{z_\beta(Q') - z_\beta(Q)}{z_\alpha(Q') - z_\alpha(Q)} = \left.\frac{dz_\beta}{dz_\alpha}\right|_{z_\alpha(Q)} .$$

In other words $\Phi_i^*\delta_i = T_S$.

(ii) Assume $i \geq 1$; then the stable curve $S^i \bigvee S$ has two components, each with just one puncture. Thus, regular 1-forms are allowed to have at worst one pole of order 1 on each component; this means (Residue Theorem) having no poles at all. So, if we choose a basis $v_1, \dots, v_i$ (resp. $v_{i+1}, \dots, v_g$) of the holomorphic forms on S^i (resp. S) the g forms together afford a trivialization of H over E_i, hence $\lambda = \bigwedge^g H$ is also trivial and so also is $\Phi_i^*\lambda$, $i \geq 1$.

To deal with the case $i = 0$, we again choose a basis $v_1, \dots, v_{g-1}$ of holomorphic 1-forms on S then, to obtain a basis of the regular 1-forms on S_Q, we add the regular form $v(Q)$, which has poles at Q and $\sigma(Q)$ with residues 1 and -1 respectively, and has real periods equal to zero [FK, p. 65].

Bers's work [Be, §4 V] proves that the form $v(Q)$ varies analytically with Q. The g-tuple $(v(Q), v_1, \dots, v_{g-1})$ is a trivialization of H over S_Q.

Note 3. Except for $i = 0$ this theorem is proved (by algebraic geometric methods) in [HM].

3.2. We now count the intersections of $\{\lambda, \delta_0, \dots, \delta_{[g/2]}\}$ with the curves $\{E, E_0, \dots, E_{[g/2]}\}$.

First of all let us examine the intersection number $\lambda \cdot E$. In order to do that, we must go back to the brief report on Satake's compactification given at the beginning of §2.

E was defined to be

$$E = \{t \in \mathbf{M_g} / F_1(\Omega_t) = \cdots = F_{3g-4}(\Omega_t) = 0\},$$

where the F_i's are certain Siegel modular forms.

On the other hand, by the definition of λ it is clear that $v_1(t) \wedge \cdots \wedge v_g(t)$ and $v_1'(t) \wedge \cdots \wedge v_g'(t)$ are nonvanishing local sections of λ. According to relation (1) of §2, these two sections are related by

$$v_1'(t) \wedge \cdots \wedge v_g'(t) = \det(C\Omega_t + D)^{-1} v_1(t) \wedge \cdots \wedge v_g(t).$$

Thus, if F is a modular form of weight 1, we see that the expression $s(t) = F(\Omega_t) v_1(t) \wedge \cdots \wedge v_g(t)$ is independent of the choice of canonical homology basis; in other words, $s(t)$ is a section of λ over $\mathbf{M_g}$, and hence over E. If more generally F has weight r, then the corresponding statement is that

$$s(t) = F(\Omega_t)(v_1(t) \wedge \cdots \wedge v_g(t))^{\otimes r}$$

is a section of the bundle $\lambda^{\otimes r}$.

Assume further that the hypersurface $\{F \equiv 0\} \subseteq \mathbf{P^n}$ meets the curve $E \subseteq \mathbf{P^n}$ in only finitely many points. Then $rE \cdot \lambda = E \cdot \lambda^{\otimes r} =$ degree $\lambda^{\otimes r}|_E$, is the degree of the divisor of the nonzero section $s(t)$ over E, i.e. the number of points (counted with multiplicities) of the intersection $E \cap \{F \equiv 0\}$.

It is a standard fact of projective geometry that the generic hypersurface $\{F \equiv 0\}$ has a nontrivial transverse intersection with E and so is that $E \cap \{F \equiv 0\}$ is never empty (see e.g. [GH, Chapter 1.3]).

Thus $E \cdot \lambda$ is nonzero. Let us call it $E \cdot \lambda = d$.

We observe that since $E \cap \{F \equiv 0\}$ is the intersection of $3g - 3$ divisors of modular forms and since modular forms are sections of (powers) of λ, then the number d is a multiple of the $(3g-3)$-fold selfintersection number of the Hodge bundle, which to our knowledge remains unknown.

Theorem 2. *For $g \geq 4$ the intersection numbers are those shown in the diagonal table below.*

		δ_0	δ_1	$\cdots$		$\delta_{[g/2]}$
E	d	0	0	$\cdots$		0
E_0	0	$\frac{2-2(g-1)}{g-1}$				
E_1	0	0	$2-2(g-1)$	0	$\cdots$	0
$\vdots$				$2-2(g-i)$		
$E_{[g/2]}$	0	0	0 $\cdots$	0	$\cdots$	$2-2(g-[g/2])$

For $g = 4$ the number $E_2 \cdot D_2$ must be divided by 2.

Proof. This follows from Theorem 1, along with the following facts: (1) for $i \neq j$ E_i does not intersect D_j, and hence δ_j is trivial over E_i, and (2) the degree of the tangent bundle over S is its Euler–Poincaré characteristic.

Finally observe that for $i \geq 1$ $\deg(\Phi_i) = 1$, that is Φ_i is injective, by our assumption that S possess no automorphisms. This holds except for the case $g = 4$ where $\deg(\Phi_2) = 2$, as points which lie in the same orbit of the group of order 2 generated by the hyperelliptic involution, map into the same element in $\overline{\mathbf{M}}_{\mathbf{g}}$.

Similarly $\deg(\Phi_0) = \operatorname{ord} \sigma = g - 1$. This completes the proof.

Corollary 1. *For* $g \geq 4$,

$\{E, E_0, \dots, E_{[g/2]}\}$ *is a basis of* $H_2(\overline{\mathbf{M}}_{\mathbf{g}}, \mathbf{Q})$,

$\{c_1(\lambda), c_1(\delta_0), \dots, c_1(\delta_{[g/2]})\}$ *is a basis of* $H^2(\overline{\mathbf{M}}_{\mathbf{g}}, \mathbf{Q})$.

Proof. By a fundamental result of Harer (see [Wo]), $H_2(\overline{\mathbf{M}}_{\mathbf{g}}, \mathbf{Q}))$ (and by duality, see [Sa], $H^2(\overline{\mathbf{M}}_{\mathbf{g}}, \mathbf{Q}))$ has rank $2 + [g/2]$.

Now our intersection matrix has determinant $d \cdot ((2 - 2(g-1))/(g-1)) \cdot (2 - 2(g-1)) \cdots (2 - 2(g - [g/2]))$ for $g > 4$ and $(1/2)\cdot$ this expression for $g = 4$. In either case the determinant is nonzero. This proves the result.

Note 4. The l.u.s. for a neighborhood of a point [S] in D_1 may be redefined so as to be the quotient of our previous one (§1.1 (viii) under the action of a biholomorphic transformation of order 2. Namely the elliptic involution of S determines such a transformation of D_1 (see [Wo, 4.3]).

These new local covers are more natural in the sense that the covering group of l.u.s. is now generically trivial. The local coordinates for a neighborhood of [S] would now be (τ, ε^2) instead of (τ, ε) (see [Wo, 4.13]) and hence the corresponding line bundle $L(D_1)$ (§1.2) would be $\delta_1^{\otimes 2}$.

Remark 1 (the case $g = 3$). We point out in conclusion that all one needs in §2 to construct E_0, is to be able to make an analytic choice of pairs of *distinct* points on a Riemann surface S of genus $g - 1$. In other words, one needs to construct a complete curve in $S \times S$-{diagonal}.

In genus 3, this can be achieved (and hence a dual basis for $H_2(\overline{\mathbf{M}}_{\mathbf{g}})$ constructed) as follows.

Let S be a Riemann surface of genus 2, admitting a holomorphic mapping $\pi: S \to T$ onto an elliptic curve $T = \mathbf{C}/\mathbf{Z} + \mathbf{Z}\tau$; then the curve

$$\{(x, y) \in S \times S/\pi(x) = \pi(y) + (1+\tau)/2\}$$

solves our problem.

Such an S is for instance, the Riemann surface with algebraic equation $y^2 = x^6 - 1$; where π is the natural projection $\pi: S \to S/\langle\sigma\rangle = T$; σ being the automorphism $\sigma(x, y) = (-x, y)$.

Remark 2. The Chern class of λ over $\overline{\mathbf{M}}_{\mathbf{g}}$ can be written down in terms of theta constants. This task has been carried out in our King's College London thesis 1987, where we obtain

$$c_1(\lambda) = \sum_{[\varepsilon]\text{even}} \frac{i}{2\pi} \partial\overline{\partial} \log|\theta[\varepsilon](0, \Omega_t)|.$$

From this expression all the aforementioned statements relative to λ become trivial.

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