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EXTENSIONS OF ISOMORPHISMS BETWEEN AFFINE ALGEBRAIC SUBVARIETIES OF k^n TO AUTOMORPHISMS OF k^n

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ABSTRACT. We derive a criterion, when an isomorphism between two closed affine algebraic subvarieties in an affine space can be extended to an automorphism of the space.

1. INTRODUCTION

Let k^n be an affine n-dimensional space over an algebraically closed field k of characteristic 0; A and B be affine algebraic subvarieties of k^n; and let $\phi: A \to B$ be an isomorphism. Then ϕ can be extended to a polynomial mapping $\Phi: k^n \to k^n$ (e.g., see [Sh]). This extension Φ is not unique, and we are interested in whether it is possible to find an extension of ϕ that is a polynomial automorphism of k^n? If A and B are isomorphic algebraic contractible curves in the complex plane, then the theorems of Abhyankar–Moh–Suzuki and Lin–Zaidenberg give the positive answer to this question [AM, S, LZ]. Z. Jelonek treated the case of smooth subvarieties A and B. His theorem says that in this case an isomorphism $A \to B$ can be extended to a polynomial automorphism if $n > 4\dim A + 1$ [J]. The main result of this paper is:

Theorem 1. *Let $\phi: A \to B$ be an isomorphism between two closed affine algebraic subvarieties A and B of k^n, and TA be the Zariski's tangent bundle of A. If*

$$n > \max(2\dim A + 1, \dim TA), \tag{1.1}$$

then ϕ can be extended to a polynomial automorphism of k^n.

In particular, when A and B are smooth subvarieties, the right-hand side of (1.1) equals $2\dim A + 1$, and Theorem 1 gives the following generalization of the Abhyankar–Moh–Suzuki's theorem.

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Corollary 2. *Let Γ be an affine algebraic curve in k^n. If Γ is isomorphic to k and $n > 3$, then there exists a polynomial automorphism that maps Γ to a coordinate axis.*

This result was also proved in [J].

We restrict ourselves to the case of the field of complex numbers C. According to "Lefschetz Principle" all other cases can be reduced to this one [BCW].

The paper contains four sections, including the introduction. In the second section we discuss several general facts from algebraic geometry. In particular, we prove that a germ of a rational function on an algebraic variety that coincides with a germ of a holomorphic function can be considered a germ of a regular function. We are not sure that it is a new fact. At least in the special case of the germs to be located at a regular point, one can find this theorem in [Sh]. Using this fact, we show that if a morphism of algebraic varieties is a biholomorphic equivalence, then it is an isomorphism. The third section contains a proof of Theorem 1. In the fourth section we construct an example, which shows that condition (1.1) cannot be improved.

The results of this paper were announced in [K]. The author is pleased to express his thanks to V. Lin, M. Stesin, and M. Zaidenberg for stimulating discussions.

2. Preliminaries

First we fix notations for this section. Let A be a local Noetherian ring, and $\overline{A}$ be its μ-adic completion, where μ is the maximal ideal of A. For every ideal α in A let $\overline{\alpha}$ be its closure in $\overline{A}$ in the μ-topology. We recall some relationships between α and $\overline{\alpha}$ (e.g., see [SZ]).

(a) The elements of α generate $\overline{\alpha}$ over the ring $\overline{A}$.

(b) $\overline{\alpha} + \overline{\beta} = \overline{\alpha + \beta}$.

(c) If α is a prime ideal then $\overline{\alpha}$ coincides with its own radical.

(d) $\bigcap_{k=1}^{\infty}(\alpha + \mu^k) = \alpha$ (the Krull's theorem).

In other words, the Krull's theorem tells that $\overline{\alpha} \cap A = \alpha$. Hence

(d$'$) If $\overline{\alpha} \subset \overline{\beta}$ then $\alpha \subset \beta$.

Denote by R the ring of germs of regular functions at the origin in C^n, and by H the ring of the germs of the holomorphic function at the origin in C^n. These rings are local, and their completions coincide with the ring F of formal power series in n complex variables.

Lemma 3. *Let I be a prime ideal in R that determines a germ V of an algebraic variety at the origin. Let J be the ideal in H generated by the germs vanishing on V. Then I generates J over the ring H.*

Proof. Let I generate ideal K over the ring H. By Nullstellensatz, for every $f \in J$ there exists a positive integer m such that $f^m \in K$ (e.g., see [GR]). By (a), $\overline{I} = \overline{K}$, and $f^m \in \overline{I}$. Then (c) implies $\overline{I} \supset J$. On the other hand, $J \supset K$. Therefore, $\overline{J} = \overline{K}$. Using (d'), we obtain $J = K$. $\square$

The following theorem is a natural corollary of Lemma 3. This theorem is not necessary for the proof of our main result, and we place it here for the sake of completeness.

Theorem 4. *Let $\widetilde{V}$ be a closed affine algebraic subvariety in C^n generated by a prime ideal $\widetilde{I}$ in the ring of polynomials on C^n. If $\widetilde{J}$ is an ideal in the ring* Hol *of holomorphic functions on C^n, which consists of the functions vanishing on $\widetilde{V}$. Then $\widetilde{I}$ generated $\widetilde{J}$ over* Hol.

Proof. Let polynomials $f_1, \dots, f_m$ be generators of $\widetilde{I}$, and $f_{1w}, \dots, f_{mw}$ be the germs of these polynomials at a point $w \in C^n$. Let V_w be the germ of V at the point w (perhaps, $V_w = \varnothing$), and let I_w be the ideal of the germs of regular functions at w that are vanishing on V_w. Clearly, $f_{1w}, \dots, f_{mw}$ are generators of I_w. By Lemma 3, for every $h \in \widetilde{J}$ and for every point $w \in C^n$, there exists a representation $h = \sum_i f_i g_i^w$ in some neighborhood U_w of w, where the functions g_i^w are holomorphic in U_w. Then $\sum_i f_i(g_i^w - g_i^u) = 0$ over $U_w \cap U_u$, i.e., $\{(g_i^w - g_i^u)_{i=1}^m\}$ is a one-cocycle with coefficients in the sheaf of relations among $f_1, \dots, f_m$. By the Oka's theorem, this sheaf is coherent (e.g., see [GR]). Hence, there exists such a cocycle $\{(s_i^w)_{i=1}^m\}$ that $\sum_i f_i s_i^w = 0$ over U^w, and $g_i^w - s_i^w = g_i^u - s_i^u$ over $U_w \cap U_u$ for each pair of points u and w. The set $\{g_i^w - s_i^w\}$ determines a global holomorphic function g_i, and $h = \sum_i f_i g_i$. □

Theorem 5. *Let V be an algebraic variety, r be a rational function on V. If r can be extended as a holomorphic function in a neighborhood of a point $w \in W$, then the function r is regular at w.*

Proof. We can assume that V is an algebraic subvariety of C^n, and w is the origin in C^n. Let $r = p/q$, where p and q are polynomials. By α_p and α_q we denote the principle ideals in R generated by p and q respectively. By α_p' and α_q' we denote the similar principle ideals in H. Under the assumptions of the theorem, there exists a germ f of a holomorphic function such that $p = qf + j$, where $j \in J$. Thus $\alpha_p' \subset \alpha_q' + J$. Lemma 3, (a) and (b) imply $\overline{\alpha}_p = \overline{\alpha}_p' \subset \overline{\alpha_q' + J} = \overline{\alpha}_q' + \overline{J} = \overline{\alpha}_q + \overline{I} = \overline{\alpha_q + I}$. By (d$'$), $\alpha_p \subset \alpha_q + I$, ie., there exists $t \in R$ and $i \in I$ such that $t = (p - i)/q$. □

Theorem 6. *Let $\phi\colon A \to B$ be a morphism of algebraic varieties A and B. If ϕ is a biholomorphic equivalence of A and B as complex spaces, then ϕ is an isomorphism.*

Proof. Since ϕ is one-to-one on the smooth parts of A and B, ϕ induces the isomorphism of the fields of the rational functions on A and B. Let U be a Zariski open affine subset of A and $\phi(U) = V$. Let $x_1, \dots, x_m$ be coordinate functions on U. Since x_k is a rational function on A, $x_k \circ \phi$ is a rational function on V. Moreover it is a holomorphic function on V, and, by Theorem 5, it is regular. Hence V is a Zariski open subset of B, and the restriction of

ϕ^{-1} to V is regular. Clearly, subsets of the type of V cover the whole variety B. Therefore ϕ^{-1} is an isomorphism. □

3. Proof of Theorem 1

Proposition 7. *Let A and B be closed affine subvarieties of C^n, and let $\phi: A \to B$ be a morphism such that*

(a) *ϕ is bijective.*

(b) *for every point $a \in A$ and $b = \phi(a)$ the induced mapping of the tangent spaces $\phi_{a*}: T_aA \to T_bB$ is isomorphism.*

(c) *ϕ is a finite morphism.*

Then ϕ is an isomorphism between A and B.

Proof. Let A_a be a germ of A at a point $a \in A$, and B_b be a germ of B at the point $b = \phi(a)$. Suppose there exists an irreducible branch D of B_b that does not belong to $\phi(A_a)$. Then we choose a sequence of points $\{b_j\}$ in $D - \phi(A_a)$ such that $b_j \to b$. Let $a_j = \phi^{-1}(b_j)$. Since $b_j \notin \phi(A_a)$, the sequence a_j cannot converge to a, and no other point in C^n is a limiting point of $\{a_j\}$ (because A is closed, and ϕ is bijective). But it does not tend to infinity because of finiteness of ϕ (e.g. see [Sh]). Hence D does not exist, and ϕ is a homeomorphism. According to Theorem 6, it is enough to verify that ϕ is biholomorphic. We choose a sufficiently small neighborhood U of a point $a \in A$. Suppose a coincides with the origin in C^n, and $T_aA = \{(x_{k+1} = \cdots = x_n = 0)\}$. Then the projector $\rho: U \to T_aA$ given by the formula $(x_1, \dots, x_n) \to (x_1, \dots, x_k)$ establishes a biholomorphic equivalence between U and $\rho(U)$. We can construct an analogous projector $\tau: V \to T_bB$ for the point $b = \phi(a)$ and $V = \phi(U)$. Let $\chi: \tau(V) \to V$ be the inverse mapping for τ. It remains to mention that the restriction ϕ to U coincides with $\chi \circ \phi_{*a} \circ \rho$. □

At this point we fix several notations. Let us consider C^{2n} as a direct sum $C^n \oplus C^n$, which is naturally embedded in $X = X_1 \times X_2$ with $X_k \cong CP^n$. We choose a coordinate system $(x, y) = (x_1, \dots, x_n, y_1, \dots, y_n)$ and the homogeneous coordinate systems $(t_{0k}, \dots, t_{nk})$ in X_k $(k = 1, 2)$ such that $x_i = t_{i1}/t_{01}$ and $y_i = t_{i2}/t_{02}$. For a subset $A \subset C^{2n}$ we denote by A_X the set $\overline{A} - A$, where $\overline{A}$ is a closure of A in X. If A is an affine algebraic subvariety in C^{2n}, then the tangent space T_aA at any point $a \in A$ has the natural embedding in the space $W \cong C^{2n}$ of the constant vector field on C^{2n}. The Zariski tangent bundle of A is the set $TA = \{(a, v) | a \in A, v \in T_aA\}$. Let T^0A be the image of TA under the mapping $(a, v) \to a + v$, where we treat $v \in W$ as a vector in C^{2n}. Recall that the chord variety CA of A is the closure in C^{2n} of the set of lines, crossing A at least at two points (since we have fixed the coordinate system, this definition is correct). Let $LA = CA \cup T^0A$. It is easy to show that LA is a closed affine algebraic subvariety in C^{2n}, when A

is the same one. We set $l(A) = \max(2\dim A + 1, \dim TA)$. It is known that $\dim CA \le 2\dim A + 1$ (e.g., see [GH]). Hence $\dim LA \le l(A)$.

Proposition 8. *Let $\phi\colon C^n \to C^{m_k}$ $(k = 1, 2)$ be linear mappings, A be a closed affine subvariety of C^{2n}, $\phi\colon A \to C^m$ $(m = m_1 + m_2)$ be the restriction of the mapping $\phi_1 \oplus \phi_2$ to A, and $V = \ker\phi_1 \oplus \ker\phi_2$. If*

$$\overline{LA} \cap V_X = \varnothing, \tag{3.1}$$

then $B = \phi(A)$ is a closed affine algebraic subvariety of C^m and $\phi\colon A \to B$ is an isomorphism.

Proof. Without loss of generality we can suppose that $m_k \le n$, and $\phi_1 \oplus \phi_2$ is given by the formula

$$(x, y) \to (x_1, \dots, x_{m_1}, y_1, \dots, y_{m_2}). \tag{3.2}$$

Then $V_X = V^1 \cup V^2$, where $V^1 = U^1 \times X_2$, $V^2 = X_1 \times U^2$, and $U^k = \{t_{0k} = \cdots = t_{m_k k} = 0\}$. Under the assumptions of the proposition, $\overline{LA} \cap V_X = \varnothing$, i.e., for every point of LA and every $k = 1, 2$ there exists at least one nonzero coordinate t_{ki} with $i \le m_k$. Since $A \subset LA$, one can define the regular mapping $\overline{\phi}\colon \overline{A} \to Y \stackrel{\text{def}}{=} CP^{m_1} \times CP^{m_2}$ by the formula

$$(t_{01}, \dots, t_{n1}, t_{02}, \dots, t_{n2}) \to (t_{01}, \dots, t_{m_1 1}, t_{02}, \dots, t_{m_2 2}).$$

Clearly, $\overline{\phi}(A_X) \subset Y - C^m$. Hence B is a closed affine algebraic subvariety in C^m. It is easy to show that (3.1) implies conditions (a) and (b) of Proposition 7 (for instance, if $\overline{CA} \cap V_X = \varnothing$ then ϕ is bijective). Thus it remains to verify that ϕ is a finite mapping. Let $N = 4(n+1)^2 - 1$, and

$$\{T_{ij}^{ks} | k, s = 1, 2;\ i, j = 0, \dots, n\}$$

be a homogeneous coordinate system in CP^N. We consider the embedding $\psi\colon X \to CP^N$ given by the formulas $\{T_{ij}^{ks} = t_{ik}t_{js}\}$. Let

$$U = \{T_{00}^{12} = T_{11}^{11} = \cdots = T_{m_1 m_1}^{11} = T_{11}^{22} = \cdots = T_{m_2 m_2}^{22} = 0\}.$$

Set $D = \psi(\overline{A})$. The spaces $\overline{A}$ and V_X are not intersecting, thus $U \cap D = \varnothing$. Hence the projector ρ from D with the center at U in a projective space $E \cong CP^m$ is a finite mapping (e.g. see [Sh]). The mapping $\overline{\chi} = \rho \circ \psi\colon \overline{A} \to \rho(D)$ is a finite mapping as well. We can use

$$\{T_{00}^{12}, T_{11}^{11}, \dots, T_{m_1 m_2}^{11}, T_{22}^{22}, \dots, T_{m_2 m_2}^{22}\}$$

as a homogeneous coordinate system in E. Let H be a hyperplane in E given by the equation $T_{00}^{12} = 0$, and $G = \rho(D) - H$. Since $\overline{\chi}^{-1}(G) = A$, the restriction of $\overline{\chi}$ to A is a finite mapping; denote it by $\chi\colon A \to G$. One can consider

$$\{T_{ii}^{kk}/T_{00}^{12} | k = 1, 2;\ i = 1, \dots, m_k\}$$

as a coordinate system in the affine space $E - H \cong C^m$. In this system the mapping χ has the following representation

$$(3.3) \qquad (x_1, \dots, x_n, y_1, \dots, y_n) \to (x_1^2, \dots, x_{m_1}^2, y_1^2, \dots, y_{m_2}^2).$$

Denote by $C[A]$, $C[B]$, and $C[G]$ the rings of regular functions over A, B, and G respectively. The mappings ϕ and χ induce the embeddings of $C[B]$ and $C[G]$ in $C[A]$. It enables us to identify the rings $C[B]$ and $C[G]$ with their images in the ring $C[A]$. Then (3.2) and (3.3) imply $C[G] \subset C[B]$. Since χ is finite, $C[A]$ is a finitely generated $C[G]$-module. Hence $C[A]$ is a finitely generated $C[B]$-module and ϕ is a finite mapping by definition. □

Proposition 9. *Let $\phi: A \to B$ be an isomorphism between closed affine algebraic subvarieties A and B in C^n. Let ϕ coincide with the restriction of a linear endomorphism $\tilde{\phi}: C^n \to C^n$ to A. Then ϕ can be extended to a polynomial automorphism of C^n.*

Proof. Without loss of generality we suppose that the formula

$$(x) \to (x_1, \dots, x_m, 0, \dots, 0)$$

gives the mapping $\tilde{\phi}$. Denote by $x' = (x_1, \dots, x_m)$ the coordinate system in the subspace $C^m \cong \{x_{m+1} = \cdots = x_n = 0\}$, which contains B. The inverse mapping ϕ^{-1} coincides with the restriction of polynomial mapping $\chi: C^m \to C^n$. Obviously, χ must be given by the formula

$$\chi(x') = (x_1, \dots, x_m, q_{m+1}(x'), \dots, q_n(x')).$$

We set

$$\alpha(x) = (x_1, \dots, x_n, x_{m+1} - q_{m+1}(x'), \dots, x_n - q_n(x')).$$

The polynomial automorphism α of C^n is what we need. □

Denote by $\rho_{m,k}: C^{2n} \to C^{k+m}$ $(0 \le m, k \le n)$ the following projector

$$(x, y) \to (x_1, \dots, x_m, y_1, \dots, y_k).$$

Definition. Let $\phi: A \to B$ be an isomorphism of closed affine algebraic subvarieties in C^n and $\Gamma \subset C^{2n}$ be its graph. We shall say the triple (ϕ, A, B) is an admissible one, if

(i) $n \ge l \stackrel{\text{def}}{=} l(A)$.
(ii) for every $m = 0, 1, \dots, l$ the set $D_m = \rho_{m,l-m}(\Gamma)$ is a closed affine algebraic subvariety in C^l.
(iii) the restriction $\rho_{m,l-m}$ to Γ is an isomorphism between Γ and D_m.

Proposition 10. *If a triple (ϕ, A, B) is admissible, then ϕ can be extended to a polynomial automorphism of C^n.*

Proof. By $p_m: C^{l+1} \to C^l$ and $q: C^{l+1} \to C^l$ we denote the projectors killing mth and $(l+1)$th coordinates respectively. Let $G_m = \rho_{m,l-m+1}(\Gamma)$. Then

$G_m \subset C^{l+1}$, $p_m(G_m) = D_{m-1}$, and $q(G_m) = D_m$. It easily follows from the assumptions of the proposition that G_m is a closed affine algebraic subvariety in C^l, and the restrictions of p_m or q to G_m are isomorphisms between G_m and D_{m-1} or D_m respectively. Let us consider C^l and C^{l+1} as linear subspaces in C^n. Then we can extend p_m and q to linear endomorphisms of C^n. We use the same notations p_m and q for these endomorphisms. By Proposition 9, there exists polynomial automorphisms β_m and γ_m such that $\beta_m|G_m = p_m|G_m$ and $\gamma_m|G_m = p_m|G_m$. Set $\mu_m = \gamma_m \circ \beta_m^{-1}$. Then $\mu_m(D_m) = D_{m-1}$, and the automorphism $\mu = \mu_1 \circ \cdots \circ \mu_l$ maps D_l in D_0. Proposition 9 shows that there exist polynomial automorphisms η and ν such that $\nu(B) = D_0$ and $\eta(A) = D_l$. Let $\alpha = \nu^{-1} \circ \mu \circ \eta$. This automorphism gives the desired extension. □

Recall that $(x, y) = (x_1, \dots, x_n, y_1, \dots, y_n)$ is a fixed coordinate system in $C^{2n} \cong C^n \oplus C^n$. By F_m $(m = 0, \dots, l)$ denote the space of linear mappings of C^{2n} in C^n with the first m coordinate functions depending on x only and the rest of them depending on y.

Proposition 11. *Let $n < l(A)$. Then there exists an algebraic subvariety N_m of codimension 1 in F_m such that for every point $P \in F_m - N_m$ the set $B = P(A)$ is a closed affine algebraic subvariety in C^l, and the restriction of P to A is an isomorphism $A \to B$.*

Proof. For every $P \in F_m$ we have

$$P(x, y) = (p_1(x), \dots, p_m(x), q_1(y), \dots, q_{l-m}(x)),$$

where p_i and q_j are linear functions. Let

$$\begin{aligned} V &= \{p_1(t_{11}, \dots, t_{n1}) = \cdots = p_m(t_{11}, \dots, t_{n1}) \\ &= q_1(t_{12}, \dots, t_{n2}) = \cdots = q_{l-m}(t_{12}, \dots, t_{n2}) = t_{01}t_{02} = 0\} \end{aligned} \tag{3.4}$$

be the subset in X (here $(t_{0k}, \dots, t_{nk})$ are the same as the beginning of this section). Let M_m be the complex space of all subvarieties $\{V\}$ given by equations of the type (3.4). Obviously, the correspondence $P \to V$ gives the natural bundle $\rho\colon F_m \to M$. By Proposition 8, it is enough to check the condition

$$\overline{LA} \cap V = \varnothing \tag{3.5}$$

for every point of M, outside a proper subvariety. The complement of C^{2n} in X is a union of E_1 and E_2, where $E_k = \{t_{0k} = 0\}$. Set $V_k = V \cap E_k$. Then $V = V_1 \cup V_2$, and we can rewrite (3.4) in such a way

$$(\overline{LA} \cap E_k) \cap V_k = \varnothing, \qquad k = 1, 2. \tag{3.6}$$

Further we restrict ourselves to the case of $k = 1$. We can consider V_1 as the product

$$V_1 = V_1^1 \times V_1^2, \tag{3.7}$$

where V_1^1 is the subspace of $CP^{n-1} = \{t_{11} : \cdots : t_{n1}\}$ given by the linear equations

$$p_1(t_{11}, \dots, t_{n1}) = \cdots = p_m(t_{11}, \dots, t_{n1}) = 0,$$

and V_1^2 is the subspace $CP^n = \{t_{02} : \cdots : t_{n2}\}$ given by the linear equations

$$q_1(t_{12}, \dots, t_{n2}) = \cdots = q_{l-m}(t_{12}, \dots, t_{n2}) = 0.$$

Thus we can identify the manifold M' of all submanifolds $\{V_1\}$ of the type (3.7) with the product of Grassmanian manifolds $\mathrm{Gr}(n-1, n-1-m) \times \mathrm{Gr}(n, n-l+m)$. For every point $a \in CP^k$ the codimension of the Shubert cycle $\{\Lambda \in \mathrm{Gr}(k, s) | a \in \Lambda\}$ is equal to $k - s$ (e.g., see [GH]). Since $\dim \overline{LA} \cap E_1 \leq l-1$, the codimension in of the subvariety $\{V_1 \in M' | V^1 \cap LA \cap E_1 \neq \varnothing\} \subset M'$ is more or equal than $m + (l-m) + (l-1) = 1$. Hence (3.6) holds for every point of M', outside a proper subvariety. □

Let W be the manifold of all the pairs of linear automorphisms of C^n.

Proposition 12. *Let $\phi\colon A \to B$ be an isomorphism between closed affine algebraic subvarieties of C^n, and $l(A) < n$. Then for every pair of linear automorphisms (α, β), outside a proper subvariety in W, the triple $(\alpha(A), \beta(B), \beta \circ \phi \circ \alpha^{-1})$ is admissible.*

Proof. Denote by F the space of linear endomorphisms of C^{2n} with the first n coordinate functions depending on x only, and the rest of them depending on y. The manifold W is the complement in F of a proper algebraic subvariety. Let $\tau\colon F \to F_m$ be the following projector

$$(p_1, \dots, p_n, q_1, \dots, q_n) \to (p_1, \dots, p_m, q_1, \dots, q_{l-m}).$$

By Proposition 11, there exists a proper algebraic subvariety $R_m \subset F_m$ such that for every $P \in F_m - R_m$ the subvariety $B = P(A)$ is closed and $P\colon A \to B$ is an isomorphism. Hence for every pair $(\alpha, \beta) \in W - \bigcup_m \tau_m^{-1}(R_m)$ the triple $(\alpha(A), \beta(B), \beta \circ \phi \circ \alpha^{-1})$ is admissible. □

Propositions 10 and 12 give the proof of Theorem 1.

4. Example

It is natural to find out, if it is possible to improve the condition (1.1). We shall show that for every $n \geq 3$ there exist isomorphic closed affine algebraic subvarieties A and B in C^n with $l(A) = n$ such that there is no polynomial automorphism which maps A to B. We shall present an example for $n = 3$. For other dimensions examples are analogous. Consider the mappings $\rho_k\colon C \to C^3$ $(k = 1, 2)$ given by the formulas

$$\rho_1\colon t \to (t^7, t^{11}, t^{13}),$$
$$\rho_2\colon t \to (t^7 + t^{14}, t^{11}, t^{13}).$$

As A and B we take the curves $\rho_1(C)$ and $\rho_2(C)$ respectively. The mapping

$$\phi\colon (x, y, z) \to (x + x^2, y, z)$$

gives a proper isomorphism (here (x, y, z) is a coordinate system in C^3). The curve A is invariant relative to an automorphism

$$\gamma_\lambda\colon (x, y, z) \to (\lambda^7 x, \lambda^{11} y, \lambda^{13} z),$$

where $\lambda \in C^*$. This automorphism generates the mapping $\tilde{\gamma}_\lambda\colon t \to \lambda t$ in the commutative diagram.

$$\begin{array}{ccc} C & \xrightarrow{\tilde{\gamma}_\lambda} & C \\ \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_1} \\ A & \xrightarrow{\gamma_\lambda} & A \end{array}$$

Assume there exists a polynomial automorphism β' such that its restriction β to A is an isomorphism between A and B. The commutative diagram

$$\begin{array}{ccc} C & \xrightarrow{\tilde{\beta}} & C \\ \downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} \\ A & \xrightarrow{\beta} & B \end{array}$$

defines the mapping $\tilde{\beta}$. Since an isomorphism maps singular points to singular points, $\tilde{\beta}(t) = \lambda t$ for some $\lambda \in C^*$. Thus for $\alpha = \beta \circ \gamma_{\lambda^{-1}}$, $\tilde{\alpha}(t) = t$, and the restriction of α to A coincides with the mapping ϕ. Therefore

$$\alpha(x, y, z) = (x + x^2 + p_1(x, y, z), y + p_2(x, y, z), z + p_3(x, y, z)),$$

where all $\{p_k\}$ vanish on A. It is easy to show that, if polynomial vanishes on A, it does not contain monomials x, x^2, y, yx, z, and zx with nonzero coefficients. Hence the Jacobian $J(\alpha)$ of the mapping α coincides with $1 + 2x + h(x, y, z)$, where the polynomial h does not contain monomial x with a nonzero coefficient. This means the Jacobian is not constant and α cannot be an automorphism.

In conclusion we would like to ask two questions. We do not know if it possible to improve the condition $n > 2\dim A + 1$ in the case of smooth subvarieties and positive $\dim A$.

We would like to find out if a smooth simply connected irreducible algebraic curve in C^3 can be mapped on a coordinate axis by a polynomial automorphism.

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