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CONTINUOUS SELECTIONS OF SOLUTION SETS TO EVOLUTION EQUATIONS

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ABSTRACT. We prove the existence of a continuous selection of the multivalued map $\xi \to \mathcal{T}(\xi)$, where $\mathcal{T}(\xi)$ is the set of all weak (resp. mild) solutions of the Cauchy problem

$$\dot{x}(t) \in Ax(t) + F(t, x(t)), \qquad x(0) = \xi,$$

assuming that F is Lipschitzian with respect to x and $-A$ is a maximal monotone map (resp. A is the infinitesimal generator of a C_0-semigroup). We also establish an analog of Michael's theorem for the solution sets of the Cauchy problem $\dot{x}(t) \in F(t, x(t))$, $x(0) = \xi$.

1. Introduction and preliminaries

The existence of a continuous map $\xi \to x_\xi$ such that x_ξ is a solution of the Cauchy problem

$$\dot{x} \in F(t, x), \qquad x(0) = \xi, \tag{1.1}$$

where F is Lipschitzian with respect to x, was proved first by Cellina in [5], under the assumption that the values of F are nonempty compact sets contained in a bounded subset of R^n. The case in which F takes nonempty closed values in R^n was considered in [6]. Some extensions to Lipschitzian F, with nonempty closed values, in a separable Banach space X were treated in [3, 7].

The purpose of this paper is to give two similar results for the Cauchy problem

$$\dot{x}(t) \in Ax(t) + F(t, x(t)), \qquad x(0) = \xi, \tag{1.2}$$

where F is Lipschitzian in x, with nonempty closed values and A satisfies assumptions of the type:

(a) $A = -B$ where B is a maximal monotone map on a separable Hilbert space; and

(b) A is the infinitesimal generator of C_0-semigroup on a separable Banach space.

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Indeed, under suitable assumptions on F, if A satisfies (a) (resp. (b)), we prove the existence of a continuous selection of the multivalued map $\xi \to \mathcal{T}(\xi)$, where $\mathcal{T}(\xi)$ is the set of all weak (resp. mild) solutions of (1.2). To do this we adapt the construction in [7] based on Filippov's approach [9]. We also establish an analog of Michael's theorem [12] for the solution sets of the Cauchy problem (1.1). For properties of the set of solutions of (1.2) see also [10, 13].

Let $T > 0$, $I = [0, T]$ and denote by $\mathcal{L}$ the σ-algebra of all Lebesgue measurable subsets of I. Let X be a real separable Banach space with the norm $\|\cdot\|$. Denote by 2^X the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X. For any subset $A \subset X$, we denote by $\mathrm{cl}\,A$ the closure of A and if $A \in 2^X$ and $x \in X$ we set $d(x, A) = \inf\{\|x - y\| : y \in A\}$. Furthermore for two closed bounded nonempty subsets A, B of X, we denote by $h(A, B)$ the Hausdorff distance from A to B, that is, $h(A, B) = \max\{\sup\{d(x, B) : x \in A\}, \sup\{d(y, A) : y \in B\}\}$.

By $C(I, X)$ (resp. $L^1(I, X)$) we mean the Banach space of all continuous (resp. Bochner integrable) functions $x: I \to X$ endowed with norm $\|x\|_\infty = \sup\{\|x(t)\| : t \in I\}$ (resp. $\|x\|_1 = \int_0^T \|x(t)\|\,dt$). $AC(I, X)$ stands for the Banach space of all absolutely continuous functions $x: I \to X$ with the norm $\|x\|_{AC} = \|x(0)\| + \|\dot{x}\|_1$).

Recall that a subset K of $L^1(I, X)$ is said to be *decomposable* [11] if for every $u, v \in K$ and $A \in \mathcal{L}$, we have $u\chi_A + v\chi_{I\setminus A} \in K$, where χ_A stands for the characteristic function of A. We denote by $\mathcal{D}$ the family of all decomposable closed nonempty subsets of $L^1(I, X)$.

Let S be a separable metric space and let $\mathcal{A}$ be a σ-algebra of subsets of S. A multivalued map $G: S \to 2^X$ is said to be *lower semicontinuous* (l.s.c.) if for every closed subset C of X the set $\{s \in S : G(s) \subset C\}$ is closed in S. The map G is said to be $\mathcal{A}$*-measurable* if for every closed subset C of X, we have that $\{s \in S : G(s) \cap C \neq \emptyset\} \in \mathcal{A}$. A function $g: S \to X$ such that $g(s) \in G(s)$ for all $s \in S$ is called a *selection* of $G(\cdot)$.

We agree to say that a multifunction $F: I \times X \to 2^X$ *satisfies* (H) if F takes nonempty closed bounded values, and

- (H_1) F is $\mathcal{L} \otimes \mathcal{B}(X)$-measurable;
- (H_2) there exists $k \in L^1(I, R)$ such that $h(F(t, x), F(t, y)) \le k(t)\|x - y\|$, for all $x, y \in X$, a.e. in I;
- (H_3) there exists $\beta \in L^1(I, R)$ such that $d(0, F(t, 0)) \le \beta(t)$, $t \in I$ a.e.

The following two lemmas are used in the sequel:

Lemma 1.1 [7, Proposition 2.1]. *Let $F^*: I \times S \to 2^X$ be $\mathcal{L} \otimes \mathcal{B}(S)$-measurable with nonempty closed values, and let $F^*(t, \cdot)$ be l.s.c. for each $t \in I$. Then the map $\xi \to G_{F^*}(\xi)$ given by*

$$G_{F^*}(\xi) = \{v \in L^1(I, X) : v(t) \in F^*(t, \xi) \text{ a.e. in } I\}$$

is l.s.c. from S *into* $\mathcal{D}$ *if and only if there exists a continuous map* $\beta: S \to L^1(I, R)$ *such that for all* $\xi \in S$, *we have* $d(0, F(t, \xi)) \leq \beta(\xi)(t)$, $t \in I$ *a.e.*

Lemma 1.2 [7, Proposition 2.2]. *Let* $G: S \to \mathcal{D}$ *be a l.s.c. multifunction, and let* $\varphi: S \to L^1(I, X)$ *and* $\psi: S \to L^1(I, R)$ *be continuous maps. If for every* $\xi \in S$ *the set*

$$H(\xi) = \operatorname{cl}\{v \in G(\xi) : \|v(t) - \varphi(\xi)(t)\| < \psi(\xi)(t), \text{ a.e. in } I\} \tag{1.3}$$

is nonempty then the map $H: S \to \mathcal{D}$ *defined by* (1.3) *admits a continuous selection.*

The second lemma is a direct consequence of Proposition 4 and Theorem 3 in [2].

In §2 we study the Cauchy problem (1.2) in the case A satisfies (a). The case of A satisfying (b) is considered in §3. In the last section we prove an extension result of Michael's type for the solution sets of (1.1).

2. $A = -B$ WHERE B IS A MAXIMAL MONOTONE MAP

In this section X is a real separable Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. B is a maximal monotone map on X with domain $D(B) = \{x \in X : Bx \neq \emptyset\}$. Recall that B is said to be *maximal monotone* on X if:

(i) for all $x_1, x_2 \in D(B)$ and all $y_1 \in Bx_1$, $y_2 \in Bx_2$, we have $\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0$; and

(ii) for every $y \in X$, there exists $x \in D(B)$ such that $x + Bx = y$.

As is well known [4] $\operatorname{cl} D(B)$ is convex, and for each $x \in D(B)$, the set Bx is closed and convex.

For $\xi \in \operatorname{cl} D(B)$ and $f \in L^1(I, X)$, we consider the Cauchy problem

$$\dot{x}(t) \in -Bx(t) + f(t), \qquad x(0) = \xi. \tag{P_f}$$

Definition 2.1. A function $x: I \to X$ is called a *strong solution* of the Cauchy problem (P_f) if it is continuous on I, absolutely continuous on every compact subset of $]0, T[$, $x(0) = \xi$, and for almost all $t \in I$, we have $x(t) \in D(B)$ and $\dot{x}(t) \in -Bx(t) + f(t)$. A function $x: I \to X$ is called a *weak solution* of the Cauchy problem (P_f) if there exist two sequence $\{f_n\}_{n \in N} \subset L^1(I, X)$ and $\{x_n\}_{n \in N} \subset C(I, X)$ such that x_n is a strong solution of (P_{f_n}), f_n converges to f in $L^1(I, X)$, and x_n converges to x in $C(I, X)$.

It is known [4, Theorem 3.4] that for each $\xi \in \operatorname{cl} D(B)$ and $f \in L^1(I, X)$, there exists a unique weak solution $x_f(\cdot, \xi)$ of the Cauchy problem (P_f). Moreover [4, Lemma 3.1] if $f, g \in L^1(I, X)$, and $x_f(\cdot, \xi)$, $x_g(\cdot, \xi)$ are weak solutions of the Cauchy problems (P_f), (P_g) then, for any $0 \leq t \leq T$, we have

$$\|x_f(t, \xi) - x_g(t, \xi)\| \leq \int_0^t \|f(u) - g(u)\| \, du. \tag{2.1}$$

Set $X_0 = \operatorname{cl} D(B)$.

Remark 2.2. The map $\xi \to x_f(\cdot, \xi)$ is continuous from X_0 to $C(I, X)$. Indeed, since

$$\frac{d}{dt}\|x_f(t, \xi_1) - x_f(t, \xi_2)\|^2$$
$$= 2\langle \dot{x}_f(t, \xi_1) - f(t) + f(t) - \dot{x}_f(t, \xi_2), x_f(t, \xi_1) - x_f(t, \xi_2)\rangle \le 0,$$

we have $\|x_f(t, \xi_1) - x_f(t, \xi_2)\| \le \|x_f(0, \xi_1) - x_f(0, \xi_2)\| = \|\xi_1 - \xi_2\|$ for all $t \in I$, which implies the continuity of the map $\xi \to x_f(\cdot, \xi)$.

Now consider the Cauchy problem

$$(P_\xi) \qquad \dot{x}(t) \in -Bx(t) + F(t, x(t)), \qquad x(0) = \xi,$$

where $F: I \times X \to 2^X$ satisfies (H) and $\xi \in X_0$.

Definition 2.3. A function $x(\cdot, \xi): I \to X$ is called a *weak solution* of the Cauchy problem (P_ξ) if there exists $f(\cdot, \xi) \in L^1(I, X)$, a selection of $F(\cdot, x(\cdot, \xi))$, such that $x(\cdot, \xi)$ is a weak solution of the Cauchy problem $(P_{f(\cdot, \xi)})$.

We denote by $\mathscr{T}(\xi)$ the set of all weak solutions of (P_ξ).

Theorem 2.4. *Let B be a maximal monotone map on X, and let $F: I \times X \to 2^X$ satisfy (H). Then there exists a function $x(\cdot, \cdot): I \times X_0 \to X$ such that*

(i) $x(\cdot, \xi) \in \mathscr{T}(\xi)$ *for every* $\xi \in X_0$; *and*
(ii) $\xi \to x(\cdot, \xi)$ *is continuous from* X_0 *to* $C(I, X)$.

Proof. Fix $\varepsilon > 0$ and set $\varepsilon_n = \varepsilon/2^{n+1}$, $n \in N$. For $\xi \in X_0$, let $x_0(\cdot, \xi): I \to X$ be the unique weak solution of the Cauchy problem

$$(P_0) \qquad \dot{x}(t) \in -Bx(t), \qquad x(0) = \xi,$$

and for k and β given by (H_2) and (H_3), define $\alpha: X_0 \to L^1(I, R)$ by

$$(2.2) \qquad \alpha(\xi)(t) = \beta(t) + k(t)\|x_0(t, \xi)\|.$$

Since, by Remark 2.2, the map $\xi \to x_0(\cdot, \xi)$ is continuous from X_0 to $C(I, X)$, from (2.2) it follows that $\alpha(\cdot)$ is continuous from X_0 to $L^1(I, R)$. Moreover, as consequence of (H_2) and (H_3), for each $\xi \in X_0$ we have:

$$(2.3) \qquad d(0, F(t, x_0(t, \xi))) \le \alpha(\xi)(t) \quad \text{a.e. in } I.$$

Define $G_0: X_0 \to 2^{L^1(I, X)}$ and $H_0: X_0 \to 2^{L^1(I, X)}$ by

$$(2.4) \qquad G_0(\xi) = \{v \in L^1(I, X) : v(t) \in F(t, x_0(t, \xi)) \text{ a.e. } t \in I\},$$
$$(2.5) \qquad H_0(\xi) = \text{cl}\{v \in G_0(\xi) : \|v(t)\| < \alpha(\xi)(t) + \varepsilon_0 \text{ a.e. } t \in I\}.$$

Clearly, by virtue of (2.3) and Lemma 1.1, $G_0(\cdot)$ is l.s.c. from X_0 into $\mathscr{D}$ and $H_0(\xi) \neq \varnothing$ for each $\xi \in X_0$. Hence, by Lemma 1.2, there exists $h_0: X_0 \to L^1(I, X)$, a continuous selection of $H_0(\cdot)$. Set $f_0(t, \xi) = h_0(\xi)(t)$. Then

$f_0(\cdot, \xi)\colon X_0 \to L^1(I, X)$ is continuous, $f_0(t, \xi) \in F(t, x_0(t, \xi))$, and $\|f_0(t, \xi)\| \le \alpha(\xi)(t) + \varepsilon_0$ for $t \in I$ a.e.

Set $m(t) = \int_0^t k(u)\,du$ and for $\xi \in X_0$, $n \ge 1$, define

(2.6)
$$\beta_n(\xi)(t) = \int_0^t \alpha(\xi)(u) \frac{[m(t) - m(u)]^{n-1}}{(n-1)!}\,du + T\left(\sum_{i=0}^{n} \varepsilon_i\right) \frac{[m(t)]^{n-1}}{(n-1)!}, \qquad t \in I.$$

Since $\alpha(\cdot)$ is continuous from X_0 to $L^1(I, R)$, by (2.6) it follows that $\beta_n(\cdot)$ also is continuous from X_0 to $L^1(I, R)$.

Let $x_1(\cdot, \xi)\colon I \to X$ be the unique weak solution of the Cauchy problem $(P_{f_0(\cdot, \xi)})$. By (2.1) we have

$$\|x_1(t, \xi) - x_0(t, \xi)\| \le \int_0^t \|f_0(u, \xi)\|\,du \le \int_0^t \alpha(\xi)(u)\,du + \varepsilon_0 T < \beta_1(\xi)(t)$$

for each $\xi \in X_0$ and $t \in I\backslash\{0\}$.

We claim that there exist two sequences $\{f_n(\cdot, \xi)\}_{n\in N}$ and $\{x_n(\cdot, \xi)\})_{n\in N}$ such that for each $n \ge 1$, the following properties are satisfied:

(a) $\xi \to f_n(\cdot, \xi)$ is continuous from X_0 into $L^1(I, X)$;
(b) $f_n(t, \xi) \in F(t, x_n(t, \xi))$ for each $\xi \in X_0$ and a.e. $t \in I$;
(c) $\|f_n(t, \xi) - f_{n-1}(t, \xi)\| \le k(t)\beta_n(\xi)(t)$ for a.e. $t \in I$; and
(d) $x_n(\cdot, \xi)$ is the unique weak solution of the Cauchy problem $(P_{f_{n-1}(\cdot, \xi)})$.

Suppose we constructed $f_1, \dots, f_n$ and $x_1, \dots, x_n$ satisfying (a)–(d). Let $x_{n+1}(\cdot, \xi)\colon I \to X$ be the unique weak solution of the Cauchy problem $(P_{f_n(\cdot, \xi)})$. Then by (2.1) and (c) for $t \in I\backslash\{0\}$ we have

(2.7)
$$\begin{aligned} \|x_{n+1}(t, \xi) - x_n(t, \xi)\| &\le \int_0^t \|f_n(u, \xi) - f_{n-1}(u, \xi)\|\,du \\ &\le \int_0^t k(u)\beta_n(\xi)(u)\,du \\ &= \int_0^t \alpha(\xi)(u) \int_u^t k(\tau)\frac{[m(t) - m(\tau)]^{n-1}}{(n-1)!}\,d\tau\,du \\ &\quad + T\left(\sum_{i=0}^{n} \varepsilon_i\right) \int_0^t k(\tau)\frac{[m(\tau)]^{n-1}}{(n-1)!}\,d\tau \\ &= \int_0^t \alpha(\xi)(u)\frac{[m(t) - m(u)]^{n}}{n!}\,du \\ &\quad + T\left(\sum_{i=0}^{n} \varepsilon_i\right) \frac{[m(t)]^{n}}{n!} < \beta_{n+1}(\xi)(t). \end{aligned}$$

Hence by (H_2),

(2.8)
$$\begin{aligned} d(f_n(t, \xi), F(t, x_{n+1}(t, \xi))) &\le k(t)\|x_{n+1}(t, \xi) - x_n(t, \xi)\| \\ &< k(t)\beta_{n+1}(\xi)(t). \end{aligned}$$

By (2.8) and Lemma 1.1, we have that the multivalued map $G_{n+1}: X_0 \to 2^{L^1(I,X)}$ defined by

$$(2.10) \qquad G_{n+1}(\xi) = \{v \in L^1(I, X) : v(t) \in F(t, x_{n+1}(t, \xi)) \text{ a.e. in } I\},$$

is l.s.c. with decomposable closed nonempty values, and by (2.8),

$$(2.11) \quad H_{n+1}(\xi) = \mathrm{cl}\{v \in G_{n+1}(\xi) : \|v(t) - f_n(t, \xi)\| < k(t)\beta_{n+1}(\xi)(t) \text{ a.e. in } I\}$$

is a nonempty set. Then by Lemma 1.2, there exists $h_{n+1}: X_0 \to L^1(I, X)$ a continuous selection of $H_{n+1}(\cdot)$. Setting $f_{n+1}(t, \xi) = h_{n+1}(\xi)(t)$ for $\xi \in X_0$, $t \in I$, we have that f_{n+1} satisfies properties (a)–(c) of our claim.

By virtue of (c) and (2.7), we have

(2.12)

$$\begin{aligned}
\|f_n(\cdot, \xi) - f_{n-1}(\cdot, \xi)\|_1 &= \int_0^T \|f_n(u, \xi) - f_{n-1}(u, \xi)\| \, du \\
&\le \int_0^T \alpha(\xi)(u) \frac{[m(T) - m(u)]^n}{n!} \, du + T\left(\sum_{i=0}^n \varepsilon_i\right) \frac{[m(T)]^n}{n!} \\
&\le \frac{[\|k\|_1]^n}{n!} (\|\alpha(\xi)\|_1 + T\varepsilon).
\end{aligned}$$

Since $\xi \to \|\alpha(\xi)\|_1$ is continuous it is locally bounded. Therefore (2.12) implies that for every $\xi \in X_0$ the sequence $(f_n(\cdot, \xi'))_{n\in N}$ satisfies the Cauchy condition uniformly with respect to ξ' on some neighborhood of ξ. Hence, if $f(\cdot, \xi)$ is the limit of $(f_n(\cdot, \xi))_{n\in N}$, then $\xi \to f(\cdot, \xi)$ is continuous from X_0 into $L^1(I, X)$.

On the other hand, using (2.7) and (2.12), we have

$$\|x_{n+1}(\cdot, \xi) - x_n(\cdot, \xi)\|_\infty \le \|f_n(\cdot, \xi) - f_{n-1}(\cdot, \xi)\|_1 \le \frac{[\|k\|_1]^n}{n!} (\|\alpha(\xi)\|_1 + T\varepsilon),$$

and so as before, $(x_n(\cdot, \xi))_{n\in N}$ is Cauchy in $C(I, X)$ locally uniformly with respect to ξ. Then, denoting its limit by $x(\cdot, \xi)$, it follows that the map $\xi \to x(\cdot, \xi)$ is continuous from X_0 to $C(I, X)$. Since $x_n(\cdot, \xi)$ converges to $x(\cdot, \xi)$ uniformly and $d(f_n(t, \xi), F(t, x(t, \xi)) \le k(t)\|x_n(t, \xi) - x(t, \xi)\|$ passing to the limit along a subsequence $(f_{n_k})_{k\in N}$ of $(f_n)_{n\in N}$ converging pointwise to f, we obtain that

$$(2.13) \qquad f(t, \xi) \in F(t, x(t, \xi)) \quad \text{for each } \xi \in X_0 \text{ and } t \in I \text{ a.e.}$$

Let $x^*(\cdot, \xi)$ be the unique weak solution of the Cauchy problem

$$(P_{f(\cdot,\xi)}) \qquad \dot{x}(t) \in -Bx(t) + f(t, \xi), \qquad x(0) = \xi.$$

By (2.1), we have

$$\|x_{n+1}(t, \xi) - x^*(t, \xi)\| \le \int_0^t \|f_n(u, \xi) - f(u, \xi)\| \, du$$

from which, letting $n \to \infty$, we get $x^*(\cdot, \xi) \equiv x(\cdot, \xi)$. Therefore $x(\cdot, \xi)$ is the weak solution of $(P_{f(\cdot,\xi)})$, and by (2.13), it follows that $x(\cdot, \xi) \in \mathscr{T}(\xi)$ for every $\xi \in X_0$. □

3. A IS THE INFINITESIMAL GENERATOR OF A C_0-SEMIGROUP

In this section X is a separable Banach space and $\{G(t) : t \geq 0\} \subset L(X, X)$ is a strongly continuous semigroup of bounded linear operators from X to X having infinitesimal generator A. Consider the Cauchy problem

$$(P_\xi) \qquad \dot{x}(t) \in Ax(t) + F(t, x(t)), \qquad x(0) = \xi,$$

where $F: I \times X \to 2^X$ is a multivalued map satisfying (H) and $\xi \in X$.

Definition 3.1. A function $x(\cdot, \xi): I \to X$ is called a *mild solution* of the Cauchy problem (P_ξ) if there exists $f(\cdot, \xi) \in L^1(I, X)$ such that

(i) $f(t, \xi) \in F(t, x(t, \xi))$ for almost all $t \in I$; and
(ii) $x(t, \xi) = G(t)\xi + \int_0^t G(t-\tau)f(\tau, \xi)\,d\tau$ for each $t \in I$.

Remark 3.2. If X is finite dimensional and $G(\cdot)$ is the identity, then every mild solution of (P_ξ) is an absolutely continuous function satisfying

$$\dot{x}(t, \xi) \in F(t, x(t, \xi)) \text{ a.e. in } I, \qquad x(0, \xi) = \xi.$$

We denote by $\mathscr{T}(\xi)$ the set of all mild solutions of (P_ξ).

Theorem 3.3. *Let A be the infinitesimal generator of a C_0-semigroup $\{G(t) : t \geq 0\}$, and let F satisfy (H). Then there exists a function $x(\cdot, \cdot): I \times X \to X$ such that*

(i) *$x(\cdot, \xi) \in \mathscr{T}(\xi)$ for every $\xi \in X$; and*
(ii) *$\xi \to x(\cdot, \xi)$ is continuous from X into $C(I, X)$.*

Proof. Let $\varepsilon > 0$ be fixed, and for $n \in N$, set $\varepsilon_n = \varepsilon/2^{n+1}$. Let $M = \sup\{\|G(t)\| : t \in I\}$, and for $\xi \in X$, define $x_0(\cdot, \xi): I \to X$ by $x_0(t, \xi) = G(t)\xi$. Since

$$\|x_0(t, \xi_1) - x_0(t, \xi_2)\| \leq \|G(t)\|\,\|\xi_1 - \xi_2\| \leq M\|\xi_1 - \xi_2\|,$$

we have that $\xi \to x_0(\cdot, \xi)$ is continuous from X to $C(I, X)$. For each $\xi \in X$, let $\alpha(\xi): I \to R$ be given by

$$\alpha(\xi)(t) = \beta(t) + k(t)\|x_0(t, \xi)\|.$$

Clearly $\alpha(\cdot)$ is continuous from X to $L^1(I, R)$. Moreover, for each $\xi \in X$,

$$d(0, F(t, x_0(t, \xi))) \leq \alpha(\xi)(t) \quad \text{for a.e. } t \in I. \tag{3.1}$$

Let $G_0: X \to 2^{L^1(I,X)}$ and $H_0: X \to 2^{L^1(I,X)}$ be defined by (2.4) and (2.5) respectively. Then as in Theorem 2.4 one finds $h_0: X \to L^1(I, X)$, a continuous selection of $H_0(\cdot)$. Set $m(t) = \int_0^t k(\tau)\,d\tau$, and for $n \geq 1$, define

$\beta_n : X \to L^1(I, R)$ by

(3.2)

$$\beta_n(\xi)(t) = M^n \int_0^t \alpha(\xi)(u) \frac{[m(t) - m(u)]^{n-1}}{(n-1)!} du + M^n T \left(\sum_{i=0}^n \varepsilon_i \right) \frac{[m(t)]^{n-1}}{(n-1)!}, \quad t \in I.$$

Set $f_0(t, \xi) = h_0(\xi)(t)$, and define

$$x_1(t, \xi) = G(t)\xi + \int_0^t G(t-\tau) f_0(\tau, \xi)\, d\tau, \qquad t \in I.$$

Then $f_0(t, \xi) \in F(t, x_0(t, \xi))$, $\|f_0(t, s)\| \le \alpha(s)(t) + \varepsilon_0$, and for $t \in I \backslash \{0\}$,

$$\begin{aligned} \|x_1(t, \xi) - x_0(t, \xi)\| &\le \int_0^t \|G(t-\tau)\| \, \|f_0(\tau, \xi)\| \, d\tau \le M \int_0^t \|f_0(\tau, \xi)\| \, d\tau \\ &\le M \int_0^t \alpha(\xi)(\tau)\, d\tau + MT\varepsilon_0 < \beta_1(\xi)(t). \end{aligned}$$

We claim that there exist two sequences $\{f_n(\cdot, \xi)\}_{n \in N}$ and $\{x_n(\cdot, \xi)\}_{n \in N}$ such that for each $n \ge 1$, the following properties are satisfied:

(a) $\xi \to f_n(\cdot, \xi)$ is continuous from X into $L^1(I, X)$;
(b) $f_n(t, \xi) \in F(t, x_n(t, \xi))$ for each $\xi \in X$ and a.e. $t \in I$;
(c) $\|f_n(t, \xi) - f_{n-1}(t, \xi)\| \le k(t)\beta_n(\xi)(t)$ for a.e. $t \in I$; and
(d) $x_{n+1}(t, \xi) = G(t)\xi + \int_0^t G(t-\tau) f_n(\tau, \xi)\, d\tau$ for $t \in I$.

Suppose that we have already constructed $f_1, \dots, f_n$ and $x_1, \dots, x_n$ satisfying (a)–(d). Define $x_{n+1}(\cdot, \xi): I \to X$ by

$$x_{n+1}(t, \xi) = G(t)\xi + \int_0^t G(t-\tau) f_n(\tau, \xi)\, d\tau, \qquad t \in I.$$

Then by (d), (c), for $t \in I \backslash \{0\}$, we have

$$\begin{aligned} \|x_{n+1}(t, \xi) - x_n(t, \xi)\| &\le \int_0^t \|G(t-u)\| \, \|f_n(u, \xi) - f_{n-1}(u, \xi)\| \, du \\ &\le M \int_0^t \|f_n(u, \xi) - f_{n-1}(u, \xi)\| \, du \le M \int_0^t k(u)\beta_n(\xi)(u)\, du \\ &= M^{n+1} \int_0^t \alpha(s)(\tau) \frac{[m(t) - m(\tau)]^n}{n!} d\tau + M^{n+1} T \left(\sum_{i=0}^n \varepsilon_i \right) \frac{[m(t)]^n}{n!} \\ &< \beta_{n+1}(\xi)(t) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} d(f_n(t, \xi), F(t, x_{n+1}(t, \xi))) &\le k(t) \|x_{n+1}(t, \xi) - x_n(t, \xi)\| \\ &< k(t)\beta_{n+1}(\xi)(t). \end{aligned} \tag{3.4}$$

Let $G_{n+1} : X \to 2^{L^1(I, X)}$, $H_{n+1} : X \to 2^{L^1(I, X)}$ be defined by (2.10), (2.11) respectively. By (3.4) and Lemma 1.1, $G_{n+1}(\cdot)$ is l.s.c. from S into $\mathscr{D}$ and

$H_{n+1}(\xi) \neq \emptyset$ for each $\xi \in X$. Hence by Lemma 1.2, there exists $h_{n+1}: X \to L^1(I, X)$ a continuous selection of $H_{n+1}(\cdot)$. Then $f_{n+1}(t, \xi) = h_{n+1}(\xi)(t)$ satisfies the properties (a)–(c) of our claim. By (c) and (3.3), it follows that

$$\begin{aligned} \|x_{n+1}(\cdot, \xi) - x_n(\cdot, \xi)\|_\infty &\leq M\|f_n(\cdot, \xi) - f_{n-1}(\cdot, \xi)\|_1 \\ &\leq \frac{[M\|k\|_1]^n}{n!}(M\|\alpha(\xi)\|_1 + MT\varepsilon). \end{aligned}$$

Therefore, $(f_n(\cdot, \xi))_{n\in N}$ and $(x_n(\cdot, \xi))_{n\in N}$ are Cauchy sequences in $L^1(I, X)$ and $C(I, X)$ respectively. Let $f(\cdot, \xi) \in L^1(I, X)$ and $x(\cdot, \xi) \in C(I, X)$ be their limits. Then as in the proof of Theorem 2.4 one can show: $\xi \to f(\cdot, \xi)$ is continuous from X into $L^1(I, X)$, $\xi \to x(\cdot, \xi)$ is continuous from X into $C(I, X)$, and for all $\xi \in X$ and almost all $t \in I$, $f(t, \xi) \in F(t, x(t, \xi))$. Passing to the limit in (d) we obtain

$$x(t, \xi) = G(t)\xi + \int_0^t G(t-\tau)f(\tau, \xi)\,d\tau \quad \text{for each } t \in I$$

completing the proof. □

4. Some additional results

In this section X is a separable Banach space, assume that $F: I \times X \to 2^X$ satisfies (H), and we consider the Cauchy problem

$$\dot{x} \in F(t, x), \qquad x(0) = \xi. \tag{4.1}$$

We denote by $\mathscr{T}(\xi)$ the set of solutions of (4.1), i.e. of all $x \in AC(I, X)$ such that $x(0) = \xi$ and $\dot{x}(t) \in F(t, x(t))$ a.e. on I. It is known (see [3, 7]) that there exists $r: X \to AC(I, X)$, a continuous selection of the multivalued map $\xi \to \mathscr{T}(\xi)$.

Proposition 4.1. *Let K be a nonempty compact convex subset of X and assume that $\mathscr{T}(K)(T) \subset K$, where $\mathscr{T}(K)(T) = \{x(T) : x \in \mathscr{T}(\xi), \xi \in K\}$. Then the boundary value problem*

$$\dot{x} \in F(t, x), \qquad x(0) = x(T) \in K, \tag{4.2}$$

admits a solution.

Proof. Let $\varphi: X \to X$ be given by $\varphi(\xi) = r(\xi)(T)$. Then φ is continuous and $\varphi(K) \subset K$, hence by Schauder's Theorem there exists $\xi_0 \in K$, a fixed point of φ. Then $r(\xi_0)(T) = \xi_0 = r(\xi_0)(0)$ and so $x = r(\xi_0)$ is a solution of (4.2). □

Results of the type considered in Proposition 4.1 were also obtained in [1, 8].

The following result is an analog of Michael's theorem in [12].

Theorem 4.2. *If Y is a closed nonempty subset of X, and $\varphi: Y \to AC(I, X)$ is a continuous map such that $\varphi(\xi) \in \mathscr{T}(\xi)$ for all $\xi \in Y$, then there exists*

$\varphi^*: X \to AC(I, X)$, a continuous extension of φ such that $\varphi^*(\xi) \in \mathscr{T}(\xi)$ for all $\xi \in X$.

Proof. Let $\mathscr{T}'(\xi) = \{\dot{x} : x \in \mathscr{T}(\xi)\}$ and set $\varphi'(\xi)(t) = \frac{d}{dt}\varphi(\xi)(t)$. Then $\varphi': Y \to L^1(I, X)$ is continuous and satisfies $\varphi'(\xi) \in \mathscr{T}'(\xi)$ for all $\xi \in Y$. By Theorem 1 in [2] there exists $\lambda: X \to L^1(I, X)$, a continuous extension of φ', and by Theorem 2 in [3] there exists a continuous map $\psi: X \times L^1(I, X) \to L^1(I, X)$ such that:

(i) $\psi(\xi, u) \in \mathscr{T}'(\xi)$ for each $u \in L^1(I, X)$; and
(ii) $\psi(\xi, u) = u$ for each $u \in \mathscr{T}'(\xi)$.

Define $\eta: X \to L^1(I, X)$ by $\eta(\xi) = \psi(\xi, \lambda(\xi))$. Then by (i), we obtain that $\eta(\xi) \in \mathscr{T}'(\xi)$ for each $\xi \in X$. Moreover, since for $\xi \in Y$ we have $\lambda(\xi) = \varphi'(\xi) \in \mathscr{T}'(\xi)$, by (ii) it follows that for all $\xi \in Y$, we have $\eta(\xi) = \psi(\xi, \varphi'(\xi)) = \varphi'(\xi)$. Therefore $\eta(\cdot)$ is a continuous extension of $\varphi'(\cdot)$ and $\eta(\xi) \in \mathscr{T}'(\xi)$ for each $\xi \in X$. Setting

$$\varphi^*(\xi)(t) = \xi + \int_0^t \eta(\xi)(\tau)\, d\tau,$$

we obtain that $\varphi^*(\cdot)$ is a continuous extension of $\varphi(\cdot)$, and $\varphi^*(\xi) \in \mathscr{T}(\xi)$ for every $\xi \in X$. □

Corollary 4.3. *Let $\xi_0, \xi_1 \in X$, $\xi_0 \neq \xi_1$, and let $x_0 \in \mathscr{T}(\xi_0)$, $x_1 \in \mathscr{T}(\xi_1)$. Then there exists a continuous map $\lambda: [0, 1] \to AC(I, X)$ such that $\lambda(0) = x_0$, $\lambda(1) = x_1$, and for $\alpha \in [0, 1]$, $\lambda(\alpha) \in \mathscr{T}(\xi_\alpha)$, where $\xi_\alpha = (1-\alpha)\xi_0 + \alpha\xi_1$.*

Proof. Let $Y = \{\xi_0, \xi_1\}$ and $\varphi: Y \to AC(I, X)$ be given by $\varphi(\xi_0) = x_0$, $\varphi(\xi_1) = x_1$. By Theorem 4.2, there exists a continuous extension $\varphi^*(\cdot)$ of $\varphi(\cdot)$ such that $\varphi^*(\xi) \in \mathscr{T}(\xi)$ for every $\xi \in X$. Then the map $\lambda: [0, 1] \to AC(I, X)$ defined by $\lambda(\alpha) = \varphi^*(\xi_\alpha)$, has the properties stated in the corollary. □

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