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THERE IS A Q-SET SPACE IN ZFC

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ABSTRACT. It is shown that there is a regular T_1-space whose every subset is a G_δ-set and yet the space is not σ-discrete.

INTRODUCTION

Let us say that a topological space X is a *Q-set space* if (a) every subset of X is a G_δ-set and; (b) X is *not* σ-discrete. Note that (a) implies that X is a T_1-space and that X is σ-discrete if and only if it is σ-closed discrete. Separable metrizable Q-set spaces are simply called Q-sets (see [7], e.g.; for separable metrizable spaces, of course, " σ-discrete" is equivalent to "countable").

It has long been known that having a Q-set is consistent with ZFC, e.g. $MA+\neg CH$ implies that every separable metrizable space of size strictly between ω and c is a Q-set [7]. On other hand it was shown that in various models of ZFC set theory, there are no "nice" Q-set spaces. First of all, CH obviously implies that there are no Q-sets. Furthermore, extending results of M. Reed [8] and R. Hansell [4], the author and H. Junnila [1] showed that under $V = L$ there are no Q-sets of character $\leq c$. They also showed [1] under $V = L$ that if every subset of a space X is a G_δ-set, then X has to be fairly close to being σ-discrete: it has to be σ-left-separated. These results made use of a method of Fleissner [2]. Related but somewhat weaker statements are true in some other models; for example, the Product Measure Extension Axiom implies that there are no Q-set spaces of weak character $< c$ [5, 3].

In light of the results mentioned above one can wonder whether $V = L$ or some large cardinal forcing implies the nonexistence of Q-set spaces. The aim of this note is to show that this is impossible: there is a ZFC example of a Q-set space. Whether such a space exists was a question of H. Junnila [5]. Our proof owes much to a technique of M. E. Rudin [9].

Our terminology and notation follows that of contemporary set theory [6, 11]; in particular, we use elementary submodels to organize otherwise lengthier induction proofs. See Rudin [10] for a direct inductive proof.

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The example

Theorem. *There is a regular T_1-space X of size c such that each subset of X is a G_δ-set, but X is not σ-discrete.*

Proof. Let us call a pair $\langle A, u\rangle$ a *control pair* if

(1) $A \in [c]^\omega$;

(2) u is a function with $\mathrm{dom}(u) \in [A]^\omega$;

(3) for every $\alpha \in \mathrm{dom}(u)$, $u(\alpha) \in [\mathscr{P}(A) \times \omega \times \omega \times 2]^{<\omega}$;

(4) $\alpha, \alpha' \in \mathrm{dom}(u)$, $\alpha \neq \alpha'$ implies $\pi_1 u(\alpha) \cap \pi_1 u(\alpha') = \varnothing$, where $\pi_1 u(\alpha) = \{B \subset A : \langle B, n, k, i\rangle \in u(\alpha)$ for some $n, k \in \omega$ and $i \in 2\}$

Let $\langle A_\beta, u_\beta\rangle_{\beta<c}$ list all control pairs mentioning each pair c times. For every $Y \in \mathscr{P}(c)$ we are going to define a function $g_Y : c \to \omega + 1$ and a sequence of functions $g_{Y,n} : c \to \omega$ such that for every $\beta < c$,

(i) $g_y(\beta) = \omega$ if and only if $\beta \in Y$;

(ii) if $\alpha < \beta$, $\langle Y \cap A_\beta, n, k, 1\rangle \in u_\beta(\alpha)$, $g_Y(\alpha) \geq n$, and $g_{Y,n}(\alpha) = k$, then $g_Y(\beta) \geq n$ and $g_{Y,n}(\beta) = k$;

(iii) if $\alpha < \beta$, $\langle Y \cap A_\beta, n, k, 0\rangle \in u_\beta(\alpha)$, and either $g_Y(\alpha) < n$ or $g_{Y,n}(\alpha) \neq k$ holds, then $g_{Y,n}(\beta) \neq k$.

The above functions naturally define sets $G_{Y,n}$ and $G^i_{Y,n,k}$, which form a subbase for the topology τ we want to construct. Before going on, it may be helpful for the reader to look up the definitions of $G_{Y,n}$, $G^i_{Y,n,k}$ and property $(*)$ below and check that

(a) condition (i) makes Y a G_δ-set;

(b) conditions (ii) and (iii) imply property $(*)$.

The control pairs make sure that our space will not be σ-discrete.

Note that both (ii) and (iii) are implications whose "if" parts we refer to later in the proof.

We construct g_Y and $g_{Y,n}$ $(n \in \omega)$ by defining the sequences $\langle g_Y(\beta)\rangle_{Y\in\mathscr{P}(c)}$ and $\langle g_{Y,n}(\beta)\rangle_{Y\in\mathscr{P}(c)}$ by induction on $\beta < c$. Suppose we are done for $\alpha < \beta$, and let $Y \in \mathscr{P}(c)$. Then $g_Y(\beta) \in \omega + 1$ and $g_{Y,n}(\beta) \in \omega$ are defined by considering several cases.

Case 1. There is no triple α, n, k satisfying the "if" part of (ii) or the "if" part of (iii). Then just make sure (i) for $g_Y(\beta)$ and define $g_{Y,n}(\beta)$ arbitrarily.

Case 2. There are α, n, k satisfying the "if" part of (ii) or the "if" part of (iii). Note that by (4), there is only one such α. Let $H = \{n \in \omega : \langle Y \cap A_\beta, n, k, i\rangle \in u_\beta(\alpha)$ for some $k \in \omega$, $i \in 2\}$. Define $g_Y(\beta)$ so that $g_Y(\beta) > \max H$ and (i) is satisfied.

Next we have to define $g_{Y,n}(\beta) \in \omega$ for every $n \in \omega$. For a given n consider two subcases.

Subcase 2.1. There is a $k^* \in \omega$ such that the "if" part of (ii) holds (with k^* in place of k). Note that there is only one such $k^* \in \omega$ since if $k^{**} \in \omega$ also satisfies (ii), then $k^* = g_{Y,n}(\alpha) = k^{**}$. Set $g_{Y,n}(\beta) = k^*(= g_{Y,n}(\alpha))$.

Subcase 2.2. There is no $k \in \omega$ such that the "if" part of (ii) holds. Then simply look at all those (finitely many) $k \in \omega$ for which the "if" part of (iii) holds and make sure that $g_{Y,n}(\beta)$ is not one of those $k \in \omega$.

This completes the definition of $g_Y(\beta)$ and $g_{Y,n}(\beta)$ $(n \in \omega)$. We have to see that (i)–(iii) hold. (i) is trivial.

Suppose that α, n, k satisfy the "if" part of (ii). Then Case 2 holds and $n \in H$, so $g_Y(\beta) > n$. Furthermore Subcase 2.1 holds with $k = k^*$ so by definition $g_{Y,n}(\beta) = k^* = k$. Suppose now that α, n, k satisfy the "if" part of (iii). Then again, Case 2 holds. If, for the given α and n, Subcase 2.2 holds, then $g_{Y,n}(\beta) \neq k$ follows from the definition of $g_{Y,n}(\beta)$ in that subcase. So assume that Subcase 2.1 holds, i.e. there is a $k^* \in \omega$ such that α, n, k^* satisfy the "if" part of (ii). In particular, $g_Y(\alpha) \geq n$ holds. Thus, since α, n, k is assumed to satisfy the "if" part of (iii), $g_{Y,n}(\alpha) \neq k$. Then $g_{Y,n}(\beta) = k^* = g_{Y,n}(\alpha) \neq k$.

Now for every $Y \in \mathscr{P}(c)$ and $n, k \in \omega$, let us set

$$G_{Y,n} = \{\beta < c : g_Y(\beta) \geq n\};$$
$$G^1_{Y,n,k} = \{\beta \in G_{Y,n} : g_{Y,n}(\beta) = k\};$$
$$G^0_{Y,n,k} = c \setminus G^1_{Y,n,k}.$$

Consider the topological space $X = \langle c, \tau \rangle$, where τ is the topology generated by $\mathscr{G} = \{G^i_{Y,n,k} : Y \in \mathscr{P}(c),\ n, k \in \omega,\ i \in 2\}$ as a subbase. Note that each $G^i_{Y,n,k}$ is clopen, so τ is zero-dimensional and thus, regular. Note that $G_{Y,n} = \bigcup_{k \in \omega} G^1_{Y,n,k} \in \tau$. Further by (i), $Y = \cap_{n \in \omega} G_{Y,n}$; thus each subset of X is a G_δ-set. This implies that X is a T_1-space.

It only remains to show that X is not σ-discrete. To see this note first that (ii) and (iii) together imply that

(*) if $\alpha < \beta$, $Y \in \mathscr{P}(c)$, $\langle Y \cap A_\beta, n, k, i \rangle \in u_\beta(\alpha)$, and $\alpha \in G^i_{Y,n,k}$, then $\beta \in G^i_{Y,n,k}$.

Now let $f : c \to \omega$ code a partition of c into ω pieces and let $h : c \to [\mathscr{P}(c) \times \omega \times \omega \times 2]^{<\omega}$ be a neighborhood assignment. (By the latter, we mean that $\alpha \in H(\alpha) = \bigcap\{G^i_{Y,n,k} : \langle Y, n, k, i \rangle \in h(\alpha)\}$ holds for every $\alpha < c$.) We are going to show that there are $\alpha < \beta$ such that $f(\alpha) = f(\beta)$ and $\beta \in H(\alpha)$. Since this holds for every f and h, X cannot be σ-discrete.

To find α and β for given f and h, let θ be big enough ($\theta = (2^c)^+$ will do), and let M be a countable elementary submodel of $H(\theta)$ such that, $f, h, \langle g_Y(\beta) \rangle_{Y \in \mathscr{P}(c),\ \beta < c}, \langle g_{Y,n}(\beta) \rangle_{Y \in \mathscr{P}(c),\ \beta < c,\ n < \omega} \in M$. Set $\overline{A} = M \cap c$, and let $\langle \overline{A}, \overline{u} \rangle$ be a control pair such that whenever $v : c \to [\mathscr{P}(c) \times \omega \times \omega \times 2]^{<\omega}$ is an infinite partial function such that $v \in M$ and $\pi_1 v(\alpha)$ $(\alpha \in \mathrm{dom}(v))$ are pairwise disjoint sets, then there is an $\alpha \in \mathrm{dom}(\overline{u}) \cap \mathrm{dom}(v)$ such that $\overline{u}(\alpha) = \{\langle Y \cap \overline{A}, n, k, i \rangle : \langle Y, n, k, i \rangle \in v(\alpha)\}$. (To see that such an $\langle \overline{A}, \overline{u} \rangle$ can be taken, let $\langle v_j \rangle_{j \in \omega}$ list all functions v as above and define $\overline{u}$ by induction on j, making sure, at the jth step, that there is a $\overline{u}(\alpha)$ as above for $v = v_j$. It is easy to make sure that $\overline{u}$ satisfies (1)–(3). (4) can be satisfied because by $\overline{A} = M \cap c$, $Y, Z \in \mathscr{P}(c) \cap M$, and $Y \neq Z$ imply $Y \cap \overline{A} \neq Z \cap \overline{A}$.)

Now choose a $\beta > \sup \overline{A}$ such that $\langle \overline{A}, \overline{u} \rangle = \langle A_\beta, u_\beta \rangle$. Consider the conjunction $\varphi(\alpha)$ of the following three statements:

(a) $f(\alpha) = f(\beta)$;

(b) $h(\alpha) \cap [(\pi_1 h(\beta) \cap M) \times \omega \times \omega \times 2] = h(\beta) \cap M$;

(c) $\langle Y, n, k, i \rangle \in h(\beta) \cap M$ implies $\alpha \in G^i_{Y,n,k}$ iff $\beta \in G^i_{Y,n,k}$.

Note that $f(\beta)$, $h(\beta) \cap M$ and $\pi_1 h(\beta) \cap M$ are elements of M. Therefore $\varphi(\alpha)$ is a first order statement with parameters from M and only α free. Because $\varphi(\beta)$ holds, there is an infinite function v such that $\mathrm{dom}(v) \subset c$, each $\alpha \in \mathrm{dom}(v)$ satisfies $\varphi(\alpha)$, $v(\alpha) = h(\alpha) \backslash h(\beta) \cap M$ $(= h(\alpha) \backslash h(\beta))$ for every $\alpha \in \mathrm{dom}(c)$, and the sets $\pi_1 v(\alpha) = \pi_1 h(\alpha) \backslash \pi_1 h(\beta) \cap M$ $(\alpha \in \mathrm{dom}(c))$ are pairwise disjoint. Since all parameters from the previous sentence are from M, we may assume that $v \in M$. So there is an $\alpha \in \mathrm{dom}(v) \cap \mathrm{dom}(\overline{u})$ with

$$\overline{u}(\alpha) = \{\langle Y \cap \overline{A}, n, k, i \rangle : \langle Y, n, k, i \rangle \in v(\alpha)\}.$$

We claim that for this pair $\alpha < \beta$, $f(\alpha) = f(\beta)$, and $\beta \in H(\alpha) = \bigcap \{G^i_{Y,n,k} : \langle Y, n, k, i \rangle \in h(\alpha)\}$ hold. $f(\alpha) = f(\beta)$ follows from (a). Let $\langle Y, n, k, i \rangle \in h(\alpha)$. Note that by $\alpha \in H(\alpha)$, $\alpha \in G^i_{Y,n,k}$. To see that $\beta \in G^i_{Y,n,k}$ we consider two cases.

Case 1. $\langle Y, n, k, i \rangle \in h(\beta) \cap M$. Then by (c), $\beta \in G^i_{Y,n,k}$.

Case 2. $\langle Y, n, k, i \rangle \in h(\alpha) \backslash h(\beta) \cap M = v(\alpha)$. Then by $\langle \overline{A}, \overline{u} \rangle = \langle A_\beta, u_\beta \rangle$, $\langle Y \cap A_\beta, n, k, i \rangle \in u_\beta(\alpha)$. Thus by $(*)$, $\beta \in G^i_{Y,n,k}$.

Concluding remarks

1. The author does not know whether there is a normal Q-set space.

2. From the proof of the theorem, it easily follows that X can be made left-separated: simply add all sets of the form $c \backslash \alpha (\alpha < c)$ to the subbase $\mathscr{G}$, and leave the rest of the proof unchanged.

3. Let us once more point out that by a result in [1], $V = L$ implies that every Q-set space is σ-left-separated (although the author cannot point out an obvious σ-left-separation in ZFC for the example in the theorem).

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