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SPIN CHARACTERISTIC CLASSES AND REDUCED $K\,\mathrm{Spin}$ GROUP OF A LOW DIMENSIONAL COMPLEX

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ABSTRACT. This note studies relations between Spin bundles, over a CW-complex of dimension ≤ 9, and their first two Spin characteristic classes. In particular by taking Spin characteristic classes, it is proved that the stable classes of Spin bundles over a manifold M with dimension ≤ 8 are in one to one correspondence with the pairs of cohomology classes $(q_1, q_2) \in H^4(M; \mathbb{Z}) \times H^8(M; \mathbb{Z})$ satisfying

$$(q_1 \cup q_2 + q_2) \bmod 3 + U_3^1 \cup (q_1 \bmod 3) \equiv 0,$$

where $U_3^1 \in H^4(M; \mathbb{Z}_3)$ is the indicated Wu-class of M.

As an application a computation is made for $\widetilde{K\ \mathrm{Spin}}(M)$, where M is an eight-dimensional manifold with understood cohomology rings over $\mathbb{Z}$, $\mathbb{Z}_2$, and $\mathbb{Z}_3$.

1. INTRODUCTION

Denote by $B\,\mathrm{Spin}$ the classifying space for the topological group $\mathrm{Spin} = \bigcup \mathrm{Spin}(n)$. Then the reduced $K\,\mathrm{Spin}$ group of a topological space X can be defined by

$$\widetilde{K\ \mathrm{Spin}}(X) = [X, B\,\mathrm{Spin}].$$

Very little is known in general about how to calculate the group for a given topological space.

E. Thomas [1] proved that there are cohomology classes $Q_i \in H^{4i}(B\,\mathrm{Spin}; \mathbb{Z})$, $i = 1, 2, \ldots$, with the property

$$H^*(B\,\mathrm{Spin}; \mathbb{Z}) = \mathbb{Z}[Q_1, Q_2, \ldots] \oplus T, \qquad 2T - 0.$$

This enabled him to define the Spin characteristic classes for the stable class of a Spin bundle ξ over a topological space X by the formula

$$Q_i(\xi) = g^* Q_i \in H^{4i}(X; \mathbb{Z}),$$

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where $g: X \to B\,\mathrm{Spin}$ is the classifying map (in the stable range) for the bundle ξ.

By means of Spin characteristic classes we define a set-valued map

$$Q_X: \widetilde{K\ \mathrm{Spin}}(X) \to H_{\mathbb{Z}}^{4,8}(X) = H^4(X;\mathbb{Z}) \times H^8(X;\mathbb{Z})$$

by

$$Q_X(\xi) = (Q_1(\xi), Q_2(\xi)), \qquad \xi \in \widetilde{K\ \mathrm{Spin}}(X).$$

As

$$Q_k(\eta \oplus \gamma) = \sum_{i+j=k} Q_i(\eta) \cup Q_j(\gamma), \qquad k \le 3$$

by [1, (1,10)], Q_X is actually a homomorphism of abelian groups if we equip $H_{\mathbb{Z}}^{4,8}(X)$ with addition

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2 + a_1 \cup a_2),$$

where $(a_i, b_i) \in H_{\mathbb{Z}}^{4,8}(X)$, $i = 1, 2$.

Naturally, in order to under the group $\widetilde{K\ \mathrm{Spin}}(X)$, one asks what the image and the kernel of Q_X are.

2. Statement of the results

Let

$$R_X: H_{\mathbb{Z}}^{4,8}(X) \to H^8(X;\mathbb{Z}_3)$$

be the map

$$R_X(a, b) = (a \cup a + b) \bmod 3 + P^1(a \bmod 3),$$

where $P^1: H^4(X;\mathbb{Z}_3) \to H^8(X;\mathbb{Z}_3)$ is the Steenrod reduced 3rd power operation. With the group structure defined on $H_{\mathbb{Z}}^{4,8}(X)$, R_X is a homomorphism, and our results can be stated as follows:

Theorem. *If X is a CW-complex with dimension ≤ 9, then*
(1) *Q_X is surjective onto* $\mathrm{Ker}\, R_X$;
(2) *Q_X is injective if ${}_3H^8(X;\mathbb{Z}) \oplus H^9(X;\mathbb{Z}_2) = 0$,*
where ${}_3G$ denotes the 3-primary component of an abelian group G.

The above theorem has the following immediate consequences:

Corollary 1. *If X is a CW-complex of dimension ≤ 9, and if ${}_3H^8(X;\mathbb{Z}) \oplus H^9(X;\mathbb{Z}_2) = 0$, then Q_X is an isomorphism onto* $\mathrm{Ker}\, R_X$.

Corollary 2. *If M is an eight-dimensional closed manifold, then the stable classes of* Spin *bundles over M are in one-to-one correspondence with the pairs $(a, b) \in H^4(M;\mathbb{Z}) \times H^8(M;\mathbb{Z})$ satisfying*

$$(a \cup a + b) \bmod 3 + U_3^1 \cup (a \bmod 3) = 0,$$

where U_3^1 is the indicated Wu class of M. In particular

$$Q_M: \widetilde{K\ \mathrm{Spin}}(M) \to H_{\mathbb{Z}}^{4,8}(M)$$

is an isomorphism if M is nonorientable.

We prove the theorem in §§3, 4. By applying the theorem, a computation for $\widetilde{K\ \mathrm{Spin}}(M)$ is made in §5, where M is an eight-dimensional manifold with understood cohomology rings over $\mathbb{Z}$, $\mathbb{Z}_2$, and $\mathbb{Z}_3$.

Remark 1. Here are two examples that show the necessity of the condition

$$ {}_3H^8(X;\mathbb{Z}) \oplus H^9(X;\mathbb{Z}_2) = 0 $$

in Theorem (2).

Example. Let $f\colon S^7 \to S^7$ be the map with degree 3, and let $X = S^7 \cup_f CS^7$ be the mapping cone of f. Then, applying the cofunctor $\widetilde{K\,\mathrm{Spin}}$ to the Puppe sequence of f yields an exact sequence:

$$ 0 = \widetilde{K\ \mathrm{Spin}}(S^7) \leftarrow \widetilde{K\ \mathrm{Spin}}(X) \leftarrow \widetilde{K\ \mathrm{Spin}}(S^8) \overset{S\wedge f}{\leftarrow} \widetilde{K\ \mathrm{Spin}}(S^8). $$

As degree $(S\wedge f\colon S^8 \to S^8) = 3$ one sees that

$$ \widetilde{K\ \mathrm{Spin}}(X) = \mathbb{Z}_3. $$

However $Q_X\colon \widetilde{K\ \mathrm{Spin}}(X) \to H^{4,8}_{\mathbb{Z}}(X)$ is trivial by the first claim of the theorem.

Example. It is straightforward that $H^{4,8}_{\mathbb{Z}}(S^9) = 0$. But the Bott periodicity theorem imples that

$$ \widetilde{K\ \mathrm{Spin}}(S^9) = \mathbb{Z}_2. $$

Remark 2. Given a topological space X, let $\widetilde{KO}(X)$ be the reduced KO group for X, and let

$$ W\colon \widetilde{KO}(X) \to H^1(X;\mathbb{Z}_2) \times H^2(X;\mathbb{Z}_2) $$

be the map $W(\xi) = (w_1(\xi),\ w_2(\xi))$, where $\xi \in \widetilde{KO}(X)$ and w_i denotes the ith Stiefel–Whitney class. There is a group structure on the set $H^1(X;\mathbb{Z}_2) \times H^2(X;\mathbb{Z}_2)$ making W a homomorphism and moreover, W fits in the exact sequence

$$ H^1(X;\mathbb{Z}_2) \times H^2(X;\mathbb{Z}_2) \overset{W}{\leftarrow} \widetilde{KO}(X) \leftarrow \operatorname{Ker} W = \widetilde{K\ \mathrm{Spin}}(X) \leftarrow 0. $$

In this sense, determining $\widetilde{K\ \mathrm{Spin}}(X)$ as well as the image of W is preliminary to computing $\widetilde{KO}(X)$. This is one of the motivations of present work, and we expect to return to this in the future.

Remark 3. We are very grateful to our referee for informing us that that the theorem can be extended a few more dimensions by the same methods used in proving: Let $R'_X\colon H^{4,8}_{\mathbb{Z}}(X) \to H^{10}(X;\mathbb{Z}_2)$ be given by $R'_X(q_1, q_2) = q_1 \cup Sq^2q_1 + Sq^4Sq^2q_1 + Sq^2q_2$. Then Q_X is surjective onto $\operatorname{Ker} R_X \cap \operatorname{Ker} R'_X$ if

$\dim X \leq 10$. Let R''_X: $\operatorname{Ker} R'_X \to H^{11}(X; \mathbb{Z}_2)/Sq^2 H^9(X; \mathbb{Z}_2)$ be given by the secondary operation coming from the relation

$$Sq^2(\iota_4 \cup Sq^2\iota_4 + Sq^4 Sq^2 \iota_4 + Sq^2 \iota_8).$$

Then it is expected that Q_X is surjective onto $\operatorname{Ker} R_X \cap \operatorname{Ker} R'_X \cap \operatorname{Ker} R''_X$ if $\dim X \leq 12$. Some statements about the injectivity of Q_X can also be made. A proof for this is given in §6.

3. Examples

In this paragraph we calculate the Spin characteristic classes for some Spin bundles, which will be needed in detecting the homotopy obstructions concerned in the proof of the theorem. The computation to be carried out are based on the following fact due to E. Thomas [1].

Lemma 1. *For a Spin bundle ξ, its Pontryagin classes $P_1(\xi)$, $P_2(\xi)$ can be expresssed in terms of Spin characteristic classes $Q_1(\xi)$, $Q_2(\xi)$ as follows*

$$P_1(\xi) = 2Q_2(\xi); \qquad P_2(\xi) = 2Q_2(\xi) + Q_1(\xi)^2. \quad \square$$

Example 1. Let ξ be the canonical quaternion line bundle over S^4. Then the Euler class $e(\xi) = v$ generates $H^4(S^4, \mathbb{Z})$ and

$$(P_1(\xi) = P_2(\xi)) = (-2v, 0),$$

[2, p.243]. So

$$(Q_1(\xi), Q_2(\xi)) = (-v, 0)$$

by Lemma 1.

Example 2. Let $\beta \in \widetilde{K\,\mathrm{Spin}}(S^8) = \widetilde{KSO}\,\mathrm{Spin}(S^8) = \mathbb{Z}$ be a generator. Then a discussion in [2,p. 244] says that

$$(P_1(\beta), P_2(\beta)) = (0, \pm 6u),$$

where u is a generator for $H^8(S^8; \mathbb{Z}) = \mathbb{Z}$.

Example 3. Let $\mathbb{C}P^4$ be the four-dimensional complex projective space and η, the canonical complex line bundle over $\mathbb{C}P^4$. Then the Euler class $e(\eta) = c$ generates the cohomology ring $H^*(\mathbb{C}P^4; \mathbb{Z})$ and $\eta \oplus \eta = 2\eta$ is a Spin bundle with

$$(P_1(2\eta), P_2(2\eta)) = (2c^2, c^4).$$

So one obtains

$$(Q_1(2\eta), Q_2(2\eta)) = (c^2, 0);$$
$$(Q_1(-2\eta), Q_2(-2\eta)) = (-c^2, c^4).$$

4. Proof of the theorem

Denote by $K(\mathbb{Z}, n)$ the Eilenberg–Mac Lane complex of type $(\mathbb{Z}, n)$, and let $\Delta: X \to X \times X$ be the diagonal map. A natural bijection

$$H_{\mathbb{Z}}^{4,8}(X) = [X, K(\mathbb{Z}, 4)] \times [X, K(\mathbb{Z}, 8)] \to [X, K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 8)]$$

is given by

$$(q_1, q_2) \to q_1 \times q_2 \circ \Delta,$$

where q_i denotes both a cohomology class and its classifying map. With this point of view we can also write (q_1, q_2) to represent both an element in $H_{\mathbb{Z}}^{4,8}(X)$ and its corresponding map $X \to K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 8)$. Then to understand the image of Q_X one asks which map

$$f: X \to K(\mathbb{Z}; 4) \times K(\mathbb{Z}, 8)$$

admits a lifting relative to $Q = (Q_1, Q_2)$,

$$B\,\mathrm{Spin} \xrightarrow{\Delta} B\,\mathrm{Spin} \times B\,\mathrm{Spin} \xrightarrow{Q_1 \times Q_2} K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 8),$$

where $Q_i \in H^{4i}(B\,\mathrm{Spin}; \mathbb{Z}) = [B\,\mathrm{Spin}, K(\mathbb{Z}, 4i)]$ is the universal Spin characteristic class.

Replace the map

$$Q: B\,\mathrm{Spin} \to K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 8)$$

up to homotopy by a fibration for which we still denote by Q. Let F be the associated fiber. By Examples 1 and 2 the homotopy exact sequence for Q yields

$$\pi_r(F) = 0 \quad \text{if } r < 7;$$
$$\pi_7(F) = \mathbb{Z}_3, \qquad \pi_8(F) = 0,$$

and

$$\pi_r(F) = \pi_r(B\,\mathrm{Spin}) \quad \text{if } r \geq 9.$$

As every sheet of the Postnikov decomposition of Q is principal, one has resolution

$$\begin{array}{ccc} B\,\mathrm{Spin} & & \\ \downarrow & & \\ E_1 & \xrightarrow{k_2} & K(Z_2, 10) \\ \downarrow & & \\ K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 8) & \xrightarrow{k_1} & K(\mathbb{Z}_3, 8) \end{array}$$

with associated k-invariants k_1, k_2. It then says that

Lemma 2. *If X is a nine-dimensional CW-complex, then*

$$\mathrm{Im}\, Q_X = \mathrm{Ker}\, k_1(X).$$

Where $k_1(-)$ denotes the natural tranformation from the cofunctor $H^{4,8}_{\mathbb{Z}}(-)$ to the cofunctor $H^8(-;\mathbb{Z}_3)$ induced by k_1.

Let $y \in H^8(K(\mathbb{Z}, 8);\mathbb{Z})$, $x \in H^4(K(\mathbb{Z}, 4);\mathbb{Z})$ be the standard generators, respectively. Let $P^1: H^4(K(\mathbb{Z}, 4);\mathbb{Z}_3) \to {}^8(K(\mathbb{Z}, 4);\mathbb{Z}_3)$ be the Steenrod reduced 3 power operation. Then as a vector space over $\mathbb{Z}_3$, $H^8(K(\mathbb{Z}, 8); Z_3)$ is generated by the single element

$$y \mod 3,$$

whilst $H^8(K(\mathbb{Z}, 4);\mathbb{Z}_3)$ is generated by the two elements

$$x^2 \mod 3\text{—decomposable},$$
$$P^1(x \mod 3)\text{—primitive}.$$

Thus k_1, which as a cohomology class lies in

$$H^8(K(Z, 4) \times K(Z, 8); Z_3) = H^8(K(Z, 4); Z_3) \oplus H^8(K(Z, 8); Z_3),$$

is defined by the relation

$$b_1 x^2 \mod 3 + b_2 P^1(x \mod 3) + b_3 y \mod 3 = 0$$

for certain $b_1, b_2, b_3 \in Z_3$.

Lemma 3. $b_1 \equiv b_2 \equiv b_3 \equiv 1 \mod 3$.

Proof. By Example 3, both $(c^2, 0)$ and $(-c^2, c^4) \in H^{4,8}_Z(CP^4)$ admit lifting relative to Q, hence the following equalities hold in $H^8(CP^4; Z_3)$:

$$(c^2, 0)^* k_1 = b_1 c^4 \mod 3 + b_2 P^1(c^2 \mod 3) = 0,$$
$$(-c^2, c^4)^* k_1 = b_1 c^4 \mod 3 + b_2 P^1(c^2 \mod 3) + b_3 c^4 \mod 3 = 0.$$

As $P^1(c^2 \mod 3) = 2c^4 \mod 3$ by [3,18.21. Theorem], one obtains

$$b_1 \equiv b_2 \equiv b_3 \mod 3.$$

On the other hand, Example 2 says that an element $(0, a) \in H^{4,8}_Z(S^8)$ admits a lifting relative to Q if and only if a is divisible by 3 in $H^8(S^8; Z)$. Hence

$$b_3 \equiv 1 \mod 3.$$

This completes the proof. □

Now we can eventually set

$$k_1 = (x^2 + y) \mod 3 + P^1(x \mod 3)$$

and therefore, for any topological space X,

$$R_X = k_1(X): H^{4,8}_Z(X) \to H^8(X; Z_3).$$

The first statement of the theorem then follows from Lemma 2.

Now we proceed to the proof of claim (2). Suppose below that X is a CW-complex having dimension ≤ 9 with

$$ {}_3H^8(X, Z) \oplus H^9(X; Z_2) = 0. $$

Assume that n is an integer sufficiently large so that the natural inclusion

$$ I: B\,\mathrm{Spin}(n) \to B\,\mathrm{Spin} $$

is a 10-homotopy equivalence.

For any $\xi \in \widetilde{K\ \mathrm{Spin}}(X)$ with $(Q_1(\xi), Q_2(\xi)) = 0$, let η be a n-dimensional Spin bundle over X representing ξ. As the composition

$$ (Q_1, 0) \cdot I: B\,\mathrm{Spin}(n) \to B\,\mathrm{Spin} \to K(Z, 4) \times K(Z, 8) $$

is a 7-homotopy equivalence, $Q_1(\eta) = 0$ implies that the principal $\mathrm{Spin}(n)$ bundle associated to η

$$ \mathrm{Spin}(n) \to E_\eta \to X $$

admits a cross section f over the 7-skeleton X^7 of X. Write by

$$ o(\eta, f) \in H^8(X; \pi_7(\mathrm{Spin}(n))) = H^8(X; Z) $$

the obstruction extending f over the 8-skeleton X^8 of X.

Similar to the proof of Lemma 1.1, (ii) in [4], one can show that the Spin characteristic class $Q_2(\eta)$ is related to $o(\eta, f)$ by the formula

$$ Q_2(\eta) = 3o(\eta, f). $$

Thus $Q_2(\eta) = 0$ together with ${}_3H^8(X; Z) = 0$ implies that $o(\eta, f) = 0$. Therefore f admits an extension $\tilde{f}$ over X^8.

Now $H^9(X; Z_2) = H^9(X; \pi_8(\mathrm{Spin}(n))) = 0$ says that the obstruction extending $\tilde{f}$ over X vanishes. So η is trivial.

This proves the second assertion of the theorem.

5. Computation for $\widetilde{K\ \mathrm{Spin}}(M^8)$

Let M be an eight-dimensional closed manifold. Then

$$ {}_3H^8(M; \mathbb{Z}) \oplus H^9(M; \mathbb{Z}_2) = 0. $$

Write $H^4(M; \mathbb{Z})$ as a direct sum of cyclic groups

$$ \bigoplus_1^p (x_i) \bigoplus_1^q (y_i) \bigoplus_1^s (z_i) \bigoplus_1^t (v_i) $$

with order $x_i = \infty$, order $y_i =$ a power of 2, order $z_i =$ a power of 3, and order $v_i =$ a power of some prime $\neq 2, 3$. Consider the homomorphisms

$$ P: {}_3H^4(M; \mathbb{Z}) = \bigoplus_1^s (z_i) \overset{\bmod 3}{\to} H^4(M; \mathbb{Z}_3) \overset{\cup U_3^1}{\to} H^8(M; \mathbb{Z}_3), $$

$$ S: {}_2H^4(M; \mathbb{Z}) = \bigoplus_1^q (y_i) \overset{\bmod 2}{\to} H^4(M; \mathbb{Z}_2) \overset{\cup U_2^4}{\to} H^8(M; \mathbb{Z}_2), $$

where $U_3^1 \in H^4(M;\mathbb{Z}_3)$, $U_2^4 \in H^4(M;\mathbb{Z}_2)$ are the indicated Wu-classes of M. If $p \neq 0$ (resp. $S \neq 0$), the generator $z_1, \dots, z_s$ (resp. $y_1, \dots, y_q$) can be chosen so that there is exactly one z_i (resp. y_i), says z_1 (resp. y_1), satisfying

$$P(z_1) \neq 0 \quad (\text{resp. } S(y_1) \neq 0).$$

Let $\xi \in \widetilde{K\ \mathrm{Spin}}(S^8) = \mathbb{Z}$ be a generator, and let $f\colon M \to S^8$ be a degree 1 (mod 2 degree 1 if M is nonorientable) map. We set

$$\xi_M = f^*(\xi).$$

Corollary 3. *As an abelian group, $\widetilde{K\ \mathrm{Spin}}(M)$ has a basis, which can be characterized by the following tables:*

(1) *If either M is orientable and $P = 0$ or M is nonorientable and $S = 0$, then*

Basis for $\widetilde{KS}(M)$	order	Q_1	Q_2
α_i, $1 \leq i \leq p$	∞	x_i	$(x_i \cup x_i) \bmod 3 + P^1(x_i \bmod 3)$
β_i, $1 \leq i \leq q$	order y_i	y_i	0
ν_i, $1 \leq i \leq s$	order z_i	z_i	0
δ_i, $1 \leq i \leq t$	order v_i	v_i	0
ξ_M	∞ if M orientable	0	$3[M]$
	2 if M nonorientable	0	$[M]$

(2) If M is orientable and $P \neq 0$, then

Basis for $\widetilde{K\ \mathrm{Spin}}(M)$	order	Q_1	Q_2
α_i, $1 \leq i \leq p$	∞	x_i	$(x_i \cup x_i) \bmod 3 + P^1(x_i \bmod 3)$
β_i, $1 \leq i \leq q$	order y_i	y_i	0
ν_i	∞	z_i	$\pm[M]$
$3\nu_1 \oplus \xi_M$ if order $z_1 > 3$	$\frac{1}{3}$ order Z_i	$3Z_1$	0
ν_1, $2 \leq i \leq s$	order z_i	z_i	0
δ_i, $1 \leq i \leq t$	order v_i	v_i	0

(3) If M is nonorientable and $S \neq 0$, then

Basis for $\widetilde{K\ \mathrm{Spin}}(M)$	order	Q_1	Q_2
α_i, $1 \leq i \leq p$	∞	x_i	$(x_i \cup x_i) \bmod 3 + P^1(x_i \bmod 3)$
β_1	2 order y_1	y_i	0
β_i, $2 \leq i \leq q$	order y_i	y_i	0
ν_i, $1 \leq i \leq s$	order z_i	z_i	0
δ_i, $1 \leq i \leq t$	order v_i	v_i	0

where [M] *is a generator of* $H^8(M;\mathbb{Z})$ *and the signs* $\pm$ *refer to* $P^1 z_1 \equiv \mp[M]$ mod 3.

Proof. The existence of the Spin bundles α_i, β, ν_i, and δ_i with indicated properties follow directly from the Theorem, and their maximality and independency can be checked by the exact sequence

$$0 \leftarrow H^4(M;\mathbb{Z}) \overset{Q_1}{\leftarrow} \widetilde{K\,\mathrm{Spin}}(M) \leftarrow \operatorname{Ker} Q_1 \leftarrow 0$$

as claimed by the Theorem. Note that the Theorem also says that

$$Q_2|\operatorname{Ker} Q_1 \colon \operatorname{Ker} Q_1 \to 3H^8(M;\mathbb{Z})$$

is an isomorphism. □

6. A proof for referee's comment

Abbreviate $K(Z,4)\times K(Z,8)$ to E_0, and let us again factor the map $Q\colon B\,\mathrm{Spin} \to E_0$ into Postnikov resolution

$$\begin{array}{ccc}
B\,\mathrm{Spin} & & \\
\downarrow & & \\
E_3 & \longrightarrow & K(\mathbb{Z},13) \\
{\scriptstyle p_3}\downarrow & & \\
E_2 & \overset{\kappa_3}{\longrightarrow} & K(Z_2,11) \\
{\scriptstyle p_2}\downarrow & & \\
E_1 & \overset{\kappa_2}{\longrightarrow} & K(\mathbb{Z}_2,10) \\
{\scriptstyle p_1}\downarrow & & \\
E_0 & \overset{\kappa_1}{\longrightarrow} & k(\mathbb{Z}_3,8)
\end{array}$$

where p_i, $i = 1, 2, 3$, is the principal bundle with assumed classifying map κ_i (κ invariant).

With coefficients in $\mathbb{Z}_2$, p_1 induces an isomorphism of cohomologies p_1^*: $H^*(E_0) \to H^*(E_1)$. So we can equally well regard κ_2 as a class in $H^{11}(E_0)$ and therefore, the principal bundle

$$\pi = p_1 \circ p_2 \colon E_2 \to E_0$$

has classifying map

$$E_0 \overset{\Delta}{\to} E_0 \times E_0 \overset{\kappa_1\times\kappa_2}{\to} K(\mathbb{Z}_3,8)\times K(\mathbb{Z}_2,10).$$

As in §4, let $y \in H^8(K(Z_3,8);\mathbb{Z}$, $x \in H^4(K(Z,4);\mathbb{Z})$ be the standard generators respectively, and put

$$\iota_8 = y \bmod 2 \in H^8(K(Z,8);\mathbf{Z}_2),\ \ \iota_4 = x \bmod 2 \in H_4(K(Z,4);\mathbb{Z}_2).$$

Then as a vector space over Z_2, $H^{10}(E_0;\mathbb{Z}_2)$ has a basis

$$\iota_4 \cup Sq^2\iota_4,\ Sq^4Sq^2\iota_4,\ Sq_2\iota_8.$$

Lemma 4. $\kappa_2 = \iota_4 \cup Sq^2\iota_4 + Sq^4Sq^2\iota_4 + Sq^2\iota_8$.

Proof. Assume $\kappa_2 = a_1\iota_4 \cup Sq^2\iota_4 + a_2Sq^4Sq^2\iota_4 + a_3Sq^2\iota_8$ for some $a_1, a_2, a_3 \in \mathbb{Z}_2$.

Let BSO be the classifying space for the stable group $SO = \bigcup_{n=2}^{\infty} SO(n)$ and ν, the universal orientable real vector bundle over BSO. Let w_i be the i th Stiefel-Whitney class of ν. Then it is known that

$$H^*(BSO; \mathbb{Z}_2) = \mathbb{Z}_2[w_2, w_3, \dots].$$

Because 2ν is a Spin bundle over BSO with

$$(Q_1(2\nu), Q_2(2\nu)) \bmod 2 = (w_4(2\nu), w_8(2\nu)) = (w_2^2, w_4^2)$$

(see [1, (1.6)]), the equality

$$a_1w_2^2 \cup Sq^2w_2^2 + a_2Sq^4Sq^2w_2^2 + a_3Sq^2w_4^2 = 0$$

holds in $H^{10}(BSO; \mathbb{Z}_2)$. It turns out that

$$(a_1 + a_2)w_2^2w_3^2 + (a_2 + a_3)w_5^2 = 0$$

by Wu's formula [2, p. 94]. So there must be

$$a_1 \equiv a_2 \equiv a_3 \mod 2.$$

Let $\alpha\colon S^9 \to S^8$ be a representation of the nontrivial element of $\pi_9(S^8) = Z_2$, and let $K = S^8 \bigcup_\alpha e^{10}$ be the mapping cone of α. Then

$$H^r(K; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } r = 0, 8, 10, \\ 0 & \text{otherwise}, \end{cases}$$

and $Sq^2\colon H^8(K; \mathbb{Z}_2) \to H^{10}(K; \mathbb{Z}_2)$ is isomorphic.

By Theorem (2) (see §2), for every n the homotopy class

$$g_n = (0, 3n) \in H_Z^{4,8}(K) = [K, E_0]$$

admits a lifting to E_1. If $a_3 \equiv 0 \bmod 2$, then g_n admits a lifting to E_2, and hence to $B\,\mathrm{Spin}$ successively.

On the other hand the Puppe sequence [3, p. 35] of α yields an exact sequence of abelian groups $Z_2 = \widetilde{KO}(S^9) \xleftarrow{\alpha^*} Z = \widetilde{KO}(S^8) \longleftarrow \widetilde{KO}(K) \longleftarrow \widetilde{KO}(S^{10})$ in which α^* is onto [5, Theorem 1.2]. This implies that g_n admits a lifting to E_2 if and only if n is even. This contradiction concludes that

$$a_3 (\equiv a_2 \equiv a_1) \equiv 1 \mod 2. \quad \square$$

We proceed now to determine $\kappa_3 \in H^{11}(E_2, \mathbb{Z}_2)$. Given a topological space X, let

$$R'_X\colon H_{\mathbb{Z}}^{4,8}(X) \to H^{10}(X; \mathbb{Z}_2)$$

be the homomorphism

$$R'_X(q_1, q_2) \equiv q_1 \cup Sq^2q_1 + Sq^4Sq^2q_1 + Sq^2q_2 \bmod 2,$$

and consider the secondary cohomology operation Φ coming from the relation

$$Sq^2(\iota_4 \cup Sq^2\iota_4 + Sq^4Sq^2\iota_4 + Sq^2\iota_8) = 0.$$

Thus Φ is defined on $\operatorname{Ker} R'_X$ and for $(q_1, q_2) \in \operatorname{Ker} R'_X$, $\Phi(q_1, q_2)$ is a coset of the subgroup $Sq^2H^9(X; \mathbb{Z}_2) \subset H^{11}(X; \mathbb{Z}_2)$.

The universal example for the operation Φ is this. Consider the principal fibration induced by κ_2

$$\begin{array}{ccc} K(\mathbb{Z}_2, 9) & \xrightarrow{j} & E \\ & & p\downarrow \\ & & E_0 \xrightarrow{\kappa_2} K(\mathbb{Z}_2, 10), \end{array}$$

where j is the inclusion of the fiber. Let $\iota_9 \in H^9(K(\mathbb{Z}_2, 9); \mathbb{Z}_2)$ be the generator. Because of Adem's relation

$$Sq^2(\iota_4 \cup Sq^2\iota_4 + Sq^4Sq^2\iota_4 + Sq^2\iota_8) = 0,$$

the Serre exact sequence for p yields a unique class $u \in H^{11}(E; \mathbb{Z}_2)$ such that

$$j^*u = Sq^2\iota_9, \ u \bmod (\operatorname{im} p^*) = u.$$

Then by the definition

Lemma 5. $\Phi(q_1, q_2) = \bigcup_g g * u \subset H^{11}(X; \mathbb{Z}_2)$, *where* $(q_1, q_2) \in \operatorname{Ker} R'_X$ *and the union is taken over all maps* $g\colon X \to E$ *with* $p \circ g = (q_1, q_2)$.

Having described the fibration $\pi = p_2 \circ p_1\colon E_2 \to E_0$ as the restriction of the product bundle

$$p_1 \times p\colon E_1 \times E \to E_0 \times E_0$$

to the diagonal, we have the induced bundle map $\widetilde{\Delta}\colon E_2 \to E_1 \times E$ over the diagonal embedding $\Delta\colon E_0 \to E_0 \times E_0$. Let $h\colon E_1 \times E \to E$ be the projection into the second factor and set u' to the composition

$$E_2 \xrightarrow{\widetilde{\Delta}} E_1 \times E \xrightarrow{h} E \xrightarrow{u} K(\mathbb{Z}_2, 11)$$

with u as that in Lemma 5.

Lemma 6. $\kappa_3 = u' \in H^{11}(E_2; \mathbb{Z}_2)$.

Proof. First observe that $h \circ \widetilde{\Delta}\colon E_2 \to E$ is a fiber preserving map over the identity of E_0, which induces isomorphisms of cohomologies over $\mathbb{Z}_2$ of both total spaces and fibers. Let F stand for the fiber of π. We have the commutative diagram

$$\begin{array}{ccccccccc} \cdots H^{10}(F) & \xrightarrow{\tau} & H^{11}(E_0) & \xrightarrow{\pi^*} & H^{11}(E_2) & \xrightarrow{i^*} & H^{11}(F) & \xrightarrow{\tau} & H^{12}(E_0) \to \cdots \\ | & & | & & | & & | & & \\ \cdots H^{10}(K(\mathbb{Z}_2, 9)) & \xrightarrow{\tau'} & H^{11}(E_0) & \xrightarrow{p^*} & H^{11}(E) & \xrightarrow{j^*} & H^{11}(K(\mathbb{Z}_2, 9)) & \xrightarrow{\tau'} & H^{12}(E_0) \cdots \end{array}$$

with exact rows the Serre exact sequences for π, p respectively, where τ, τ' are the transgressions. It says that the class u' is characterized uniquely by

$$i^*u' = Sq^2\iota_9, \qquad u' \bmod (\operatorname{im} \pi^*) = u',$$

and that $H^{11}(E_2)$ is generated by the elements

$$\pi^*(\iota_4 \cup Sq^3\iota_4), \qquad \pi^*(Sq^5Sq^2\iota_4), \qquad \pi^*(Sq^3\iota_8), u'$$

subject to the single relation

$$\pi^*(\iota_4 \cup Sq^3\iota_4 + Sq^5Sq^2\iota_4 + Sq^3\iota_8) = 0,$$

because

$$\tau(Sqi_9) = Sq\kappa_2 = \iota_4 \cup Sq^4\iota_4 + Sq^5Sq^2\iota_4 + Sq^3\iota_8$$

in $H^{11}(E_0)$.

Next consider the homomorphism induced by I,

$$I^*: H^*(E_2) \to H^*(B\,\mathrm{Spin}),$$

and recall that if $\rho: B\,\mathrm{Spin} \to BSO$ is the map induced by the universal covering $\mathrm{Spin} \to SO$ and if we let $w_k^* = \rho^* w_k$ with $w_k \in H^*(BSO; \mathbb{Z}_2)$ as before, and if k is not of the form $2^r + 1 (r \geq 1)$, then

$$H^*(B\,\mathrm{Spin}; \mathbb{Z}_2) = \mathbb{Z}_2[w_4^*, w_6^*, w_7^*, \cdots]$$

whilst

$$Q_k \bmod 2 = w_{4k}^*$$

whenever $k \leq 15$ [1, (1.1), (1.6), and (1.8)]. Thus by Wu's formula [2, p. 94]

$$I^*\pi^*(\iota_4 \cup Sq^3\iota_4) = w_4^* \cup Sq^3 w_4^* = w_4^* \cup w_7^*,$$
$$I^*\pi^*(Sq^5Sq^2\iota_4) = Sq^5Sq^2 w_4^* = w_4^* \cup w_7^* + w_{11}^*,$$
$$I^*\pi^*(Sq^3\iota_8) = Sq^3 w_8^* = w_{11}^*.$$

This means that $\mathrm{Ker}[I^*: H^{11}(E_2) \to H^{11}(B\,\mathrm{Spin})] = \mathbb{Z}_2$ is spanned by u'. So there must be

$$\kappa_3 = u'. \quad \square$$

Let X be a *CW-complex* and let $R_X'': \mathrm{Ker}\, R_X' \to H^{11}(X; \mathbb{Z}_2)/Sq^2 H^9(X; \mathbb{Z}_2)$ be given by

$$R_X''(q_1, q_2) = \Phi(q_1, q_2) \mod Sq^2 H^9(X; \mathbb{Z}_2)$$

with Φ as in Lemma 5. Summarizing we have proved

Theorem 1. *If* $\dim X \leq 10$, *then*

$$Q_X: k\,\mathrm{Spin}(X) \to H_Z^{4,8}(X)$$

is surjective onto $\mathrm{Ker}\, R_X \cap \mathrm{Ker}\, R_X'$ *and if* $\dim X \leq 12$, Q_X *is surjective onto* $\mathrm{Ker}\, R_X \cap \mathrm{Ker}\, R_X' \cap \mathrm{Ker}\, R_X''$.

Moreover by the Bott periodicity theorem, the standard method in the proof of Theorem, (2) is valid to show that

Theorem 2. *If* $\dim X \leq 11$ *and if* $H^8(X; \mathbb{Z}) \oplus H^9(X; \mathbb{Z}_2) \oplus H^{10}(X; \mathbb{Z}_2) = 0$, *then* Q_X *is an isomorphism onto* $\mathrm{Ker}\, R_X \cap \mathrm{Ker}\, R_X' \cap \mathrm{Ker}\, R_X''$.

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