

MANIFOLDS WITH FINITE FIRST HOMOLOGY AS CODIMENSION 2 FIBRATORS

ROBERT J. DAVERMAN

(Communicated by James E. West)

ABSTRACT. Given a map $f: M \rightarrow B$ defined on an orientable $(n+2)$ -manifold with all point inverses having the homotopy type of a specified closed n -manifold N , we seek to catalog the manifolds N for which f is always an approximate fibration. Assuming $H_1(N)$ finite, we deduce that the cohomology sheaf of f is locally constant provided N admits no self-map of degree $d > 1$ when $H_1(N)$ has a cyclic subgroup of order d . For manifolds N possessing additional features, we achieve the approximate fibration conclusion.

1. INTRODUCTION

Codimension 2 fibrators, a notion introduced in [Da], afford quick recognition of certain maps $p: M \rightarrow B$ as approximate fibrations. This, in turn, gives rise to reasonably computable structural information, based on a long exact homotopy sequence for approximate fibrations, developed by Coram and Duvall [CD1], analogous to the classical one for fibrations. Known codimension 2 fibrators include all simply connected closed manifolds, closed 2-manifolds of negative Euler characteristic, and P^n (real projective n -space); nonfibrators in codimension 2 include all closed manifolds admitting a fixed point free cyclic action having orbit space homotopy equivalent to itself [Da].

A closed, connected manifold N^n is called a *codimension 2 (orientable) fibror* if whenever $f: M \rightarrow B$ is a proper closed map defined on an (orientable) $(n+2)$ -manifold and each $f^{-1}(b)$ is a compact ANR with the homotopy type of N^n , f is an approximate fibration (meaning, f satisfies the approximate homotopy lifting property; see [CD1]). In this setting, the upper semicontinuous (henceforth, abbreviated as usc) decomposition $\{f^{-1}(b): b \in B\}$ of M induced by f is called *N^n -like*. Instead of the ANR hypothesis, one could just as well require all $f^{-1}(b)$ in an N^n -like decomposition to be compact sets shape equivalent to N^n .

Received by the editors February 8, 1989 and, in revised form, September 5, 1989.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 55R65, 57N15, 57N65; Secondary 55M25, 57N12, 54B15.

Key words and phrases. Approximate fibration, approximate lifting, continuity set, codimension 2 fibror, geometric structure, Hopfain group, mapping degree.

Research supported in part by a grant from the National Science Foundation.

By the *degree* of a map between closed, connected, orientable n -manifolds, we mean more precisely the absolute degree; namely, the nonnegative integer determining the induced homomorphism's effect on n th (integral) homology, up to sign. Let N^n be a closed orientable n -manifold and G an N^n -like decomposition of an orientable $(n + 2)$ -manifold M . Then the decomposition space $B = M/G$ is a 2-manifold [DW]. Further analysis depends on the concept of the *continuity set* $C \subset B$ of the decomposition map $p: M \rightarrow B$, which we briefly describe now in terms of degree: each $g \in G$ has a neighborhood U_g equipped with a retraction $R_g: U_g \rightarrow g$, and $p(g) \in C$ iff R_g restricts to a degree 1 map $g' \rightarrow g$ for all $g' \in G$ sufficiently close to g . By [DW] C contains all but a locally finite subset of B .

Recall that a group Γ is *Hopfian* if every epimorphism $\Gamma \rightarrow \Gamma$ is necessarily 1 - 1.

In lieu of additional review, we restate two basic facts from [Da] in a form readily applicable here.

Proposition AF. *Suppose N^n is a closed orientable n -manifold such that either $\pi_1(N^n)$ is finite or it is Hopfian and N^n is aspherical, and suppose G is an N^n -like usc decomposition of an orientable $(n + 2)$ -manifold M . Then the decomposition map $p: M \rightarrow B$ restricts to an approximate fibration $p|_{p^{-1}(C)}: p^{-1}(C) \rightarrow C$ over the continuity set C of p .*

We close by outlining the strategy. Under the hypothesis that $H_1(N^n)$ is finite, we seek to prove that the continuity sets C of $p: M \rightarrow B$ associated with various N^n -like decompositions satisfy $C = B$. The homological finiteness condition alone does not force N^n to be a codimension 2 fibrator— $P^{2k+1} \# P^{2k+1}$ is a counterexample [Da]. Nevertheless, then all R_g restrict to maps $g' \rightarrow g$ with nonzero degree, useful data in itself, and the first result in the next section limits the possible degrees d to the orders of cyclic subgroups of $H_1(N^n)$. Of course, $C = B$ when $d = 1$ is the only possibility. Finally, we impose further conditions on N^n , like those in the hypothesis of Proposition AF, to detect approximate fibrations.

2. MANIFOLDS WITH FINITE FIRST HOMOLOGY

Proposition 1. *Suppose N^n is a closed, orientable n -manifold whose first (integral) homology group is finite. Suppose M is any orientable $(n + 2)$ -manifold, G is an N^n -like usc decomposition of M , $g_0 \in G$, and $R: U \rightarrow g_0$ is a retraction defined on a neighborhood U of g_0 . Then there exists an integer $d \geq 1$ such that*

- (1) g_0 has a neighborhood $U' \subset U$ where, for all $g \in G$ in U' , $g \neq g_0$, $R|_g: g \rightarrow g_0$ has absolute degree d ;
- (2) $H_1(N^n)$ contains a cyclic subgroup of order d .

Proof. At the core of the argument, which mostly involves algebraic diagram-chasing, is the simplification stemming from the relationships $H_{n-1}(N^n) \cong H^1(N^n) \cong 0$ insured by duality and universal coefficients.

Consider the decomposition map $p: M \rightarrow B = M/G$. In this setting B is a 2-manifold [DW]. Restricting M , we reduce to the situation where B is an open 2-cell containing $b_0 = p(g_0)$ and $p: M \rightarrow B$ is an approximate fibration over $B \setminus b_0$.

Application of approximate lifting over $B \setminus b_0$ shows M deformation retracts to g_0 , with $R: M \rightarrow g_0$ being the end of the deformation. Accordingly, $R_*: H_*(M) \rightarrow H_*(g_0)$ is an isomorphism. From the exact sequence of the pair $(M, M \setminus g_0)$,

$$0 \cong H_{n+2}(M) \rightarrow H_{n+2}(M, M \setminus g_0) \rightarrow H_{n+1}(M \setminus g_0) \rightarrow H_{n+1}(M) \cong 0,$$

yielding $H_{n+1}(M \setminus g_0) \cong Z$, and

$$0 \cong H^1(g_0) \cong H_{n+1}(M, M \setminus g_0) \rightarrow H_n(M \setminus g_0) \rightarrow H_n(M) \cong Z.$$

Finiteness of $H_1(N^n)$ is crucial to proving $H_n(M \setminus g_0)$ is the (isomorphic) image of $H_n(g)$ for any $g \in G \setminus \{g_0\}$. Express $M \setminus g_0$ as the union of G -saturated open sets V, V' whose intersection consists of components W, W' , where $p(V), p(V'), p(W), p(W')$ are all 2-cells and $g \subset W$. Since

$$H_n(V) \cong H_n(W) \cong H_n(g) \cong Z \cong H_n(V') \cong H_n(W'),$$

$$H_{n-1}(W) \cong H_{n-1}(g) \cong 0 \cong H_{n-1}(W'),$$

the Mayer-Vietoris sequence for the triad $(M \setminus g_0, V, V')$ shows

$$(\zeta) \quad 0 \rightarrow Z \cong H_{n+1}(M \setminus g_0) \rightarrow Z \oplus Z \xrightarrow{\psi_*} Z \oplus Z \rightarrow H_n(M \setminus g_0) \rightarrow 0.$$

Focus on the homomorphism $H_n(W) \oplus H_n(W') \rightarrow H_n(V) \oplus H_n(V')$ giving rise to the homomorphism $\psi_*: Z \oplus Z \rightarrow Z \oplus Z$ in (ζ) , and arrange notation to make $\psi_*(\langle s, t \rangle) = \langle s+t, s \pm t \rangle$. Sequence (ζ) shows $\ker \psi_* \neq 0$, so $\psi_*(\langle s, t \rangle) = \langle s+t, s+t \rangle$ and, thus, $H_n(M \setminus g_0) \cong (Z \oplus Z)/\text{im}(\psi_*)$ corresponds to $H_n(V) \oplus 0 \cong H_n(g) \oplus 0$.

Now examine the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & H_n(M \setminus g_0) & \xrightarrow{i_*} & H_n(M) & \xrightarrow{j_*} & H_n(M, M \setminus g_0) \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ & & Z & & Z & & H^2(g_0) \cong \text{Free} \oplus H_1(g_0). \end{array}$$

Up to sign, i_* amounts to multiplication by an integer $d \geq 1$, and j_* injects the cokernel, a cyclic subgroup of order d , in $H_1(g_0) \cong H_1(N^n)$. Finally, $R|g: g \rightarrow g_0$ has degree d , since the induced homomorphism on n th homology can be expressed as the composition of isomorphisms with i_* .

Corollary 2. *Suppose N^n is a closed, orientable, n -manifold such that $H_1(N^n)$ is finite and whenever $H_1(N^n)$ contains a cyclic subgroup of order $d > 1$, then*

N^n admits no self-map of degree d . Suppose G is an N^n -like usc decomposition of an orientable $(n + 2)$ -manifold M . Then the (integral) cohomology sheaf of the decomposition map $f: M \rightarrow B = M/G$ is locally constant.

Proof. Clearly the maps $R|g: g_0$ induce isomorphisms on n th homology. By Poincaré duality and naturality of cap products, they have the same effect on homology groups at all levels (application of 5.6.16 from [Sp, p. 254] reveals that these $(R|g)^*$ provide epimorphisms and hence isomorphisms between $(n - q)$ th homology groups). Duality implies these $(R|g)^*$ are cohomology isomorphisms in all dimensions, from which the corollary follows.

Propositions 1 and AF combine as:

Theorem 3. *Suppose N^n is a closed, orientable, n -manifold for which either $\pi_1(N^n)$ is finite or $\pi_1(N^n)$ is Hopfian, $H_1(N^n)$ is finite, and N^n is aspherical. Suppose further that if $H_1(N^n)$ contains a cyclic subgroup of order $d > 1$ then N^n admits no self-map of degree d . Then N^n is a codimension 2 orientable fibrator.*

Corollary 4. *If Γ is a finite group admitting no homomorphisms $\psi: \Gamma \rightarrow \Gamma$ with index $[\Gamma, \psi(\Gamma)] > 1$ equal to the order of a cyclic subgroup of the abelianization, $\Gamma/\text{comm}(\Gamma)$, then every orientable manifold N with $\pi_1(N) \cong \Gamma$ is an orientable codimension 2 fibrator.*

Proof. Given N with fundamental group Γ , consider any map $f: N \rightarrow N$ of degree $d > 1$, where d is the order of a cyclic subgroup of $H_1(N) \cong \Gamma/\text{comm}(\Gamma)$. Suppose $f_*(\Gamma) \neq \Gamma$. Specify a covering space $q: N^* \rightarrow N$ corresponding to $f_*(\Gamma)$ and a lift $f^*: N \rightarrow N^*$ of f . Since

$$d = \text{deg}(f) = \text{deg}(q) \cdot \text{deg}(f^*),$$

we see that $[\Gamma, f_*(\Gamma)] = \text{deg}(q)$ divides d , implying the existence of another cyclic subgroup in $H_1(N)$ of order $[\Gamma, f_*(\Gamma)]$, contrary to hypothesis. Thus, all degree d maps between copies of N encountered in our setting induce π_1 -isomorphisms, in which case we can imitate the proof of [Da, Proposition 2.6] to conclude N is a codimension 2 fibrator.

For examples, such familiar objects as the symmetric group \mathcal{S}_m on $m > 4$ symbols, the nonabelian group of order 21, and the dihedral groups

$$D_{2^m \cdot (2k+1)} = \langle x, y | x^{2^m} = 1 = y^{2k+1}, xyx^{-1} = y^{-1} \rangle$$

all satisfy the hypothesis of Corollary 4. Inspecting the third example more closely, we see it abelianizes to the cyclic group of order 2^m and any subgroup H of index 2^s is cyclic (because y commutes with x^2) of order divisible by $2k + 1$, but obviously then this dihedral group does not surject to H .

Finite abelian groups are not amenable to this analysis.

3. APPLICATIONS TO 3-MANIFOLDS

We point out another fact closely related to Proposition AF.

Proposition 5. *Suppose N^n is a closed orientable n -manifold, $n \in \{3, 4\}$, with $\pi_1(N^n)$ Hopfian and suppose G is an N^n -like usc decomposition of an orientable $(n + 2)$ -manifold M . Then the decomposition map $p: M \rightarrow B$ restricts to an approximate fibration $p|_{p^{-1}(C)}$ over the continuity set C of p .*

Proof. Hausmann [Ha, Proposition 1(b)] establishes that, over C , the retractions $R_g: U_g \rightarrow g$ restrict to homotopy equivalences $g' \rightarrow g$ for $g' \in G$ sufficiently close to g . In the context at hand this property yields the conclusion [CD2, Corollary 3.4].

Conceivably the class of manifolds N^3 described below is all-inclusive. Under this hypothesis Hempel [He2, Corollary 1.2] proves $\pi_1(N^3)$ is Hopfian (also see [He1] for the revelant definition).

Corollary 6. *Suppose N^3 is a closed, orientable 3-manifold whose prime factors are either virtually Haken or else have finite or cyclic fundamental groups, and suppose G is an N^3 -like usc decomposition of an orientable 5-manifold M . Then the decomposition map $p: M \rightarrow B$ restricts to an approximate fibration over its continuity set.*

Corollary 7. *Every closed hyperbolic 3-manifold with finite first homology is a codimension 2 fibrator.*

Proof. According to Gromov [Gr, p. 147], there is no degree $d > 1$ self-map on a closed hyperbolic manifold. (Lemma 4.1 of [He2] certifies the Hopfian part needed in Proposition 5.)

The conclusion of Corollary 7 might be valid without the H_1 -finiteness condition.

Let L^3 denote the unique closed 3-manifold having Euclidean structure and finite first homology (in other words, L^3 does not fiber over S^1). It is the orientable Seifert bundle over P^2 with Euclidean structure appearing in Scott's list [Sc, p. 446]. Standard calculations from the associated Seifert bundle data yield $H_1(L^3) \cong Z_4 \oplus Z_4$.

Lemma 8. *There is no map $L^3 \rightarrow L^3$ of degree 2 or 4.*

Proof. Suppose $f: L^3 \rightarrow L^3$ is a map of degree 2 or 4. Keep in mind that $f_*(\pi_1(L^3)) \neq \pi_1(L^3)$ for otherwise f_* would be an isomorphism and, as a self-map on a $K(\pi, 1)$ space, f would be a homotopy equivalence and, therefore, have degree 1. As in the proof of Corollary 4, take a (non-trivial) covering $q: N^* \rightarrow L^3$ corresponding to $f_*(\pi_1(L^3))$ and lift f to $f^*: L^3 \rightarrow N^*$. Significant features are: f^* induces an epimorphism at the H_1 level (of course, on π_1 as well), and q has degree 2 or 4.

We can regard L^3 as the result of identifying two copies T, T' of the orientable twisted I -bundle over a Klein bottle via the homeomorphism θ on

the boundary torus interchanging the S^1 -factors, in some preferred parameterization. (Scott [Sc, p. 448] relates that closed Euclidean 3-manifolds arise either as torus or Klein bottles bundles over S^1 or as the union of two twisted I -bundles, and the finiteness of $H_1(L^3)$ precludes L^3 from being among the former.) Thus, N^* is a union of finite-sheeted covers of T , T' , which necessarily are either I -bundles over the torus or homeomorphic lifts of T and T' themselves.

Suppose q is 2-sheeted. Then N^* could be a torus bundle over S^1 (with monodromy θ); this occurs iff both twisted I -bundles lift to $S^1 \times S^1 \times I$. Otherwise N^* is another union of twisted I -bundles over a Klein bottle. The latter occurs when (precisely) one of T , T' is covered by two copies of itself and N^* coincides with their union plus a boundary collar; a check of the various lifts shows the attaching homeomorphism to be the identity. Neither case can arise in the present context, since each requires $H_1(N^*)$ to be infinite.

Moreover, q cannot be 4-sheeted either, for it follows easily that q would factor through a 2-sheeted cover, again implying $H_1(N^*)$ is infinite.

Corollary 9. L^3 is an orientable codimension 2 fibrator.

Among closed 3-manifolds having Euclidean geometric structure, L^3 is the solitary example satisfying the conclusion of Corollary 9. Like all Euclidean manifolds, L^3 covers itself—in fact, it does so k times for every odd integer k (e.g., the cover corresponding to the subgroup generated by x^k , y^2 in the presentation for $\pi_1(L^3)$ as

$$\langle x, y | xy^2 = y^{-2}x, yx^2 = x^{-2}y \rangle,$$

where x , y^2 generate the contribution of one of these twisted I -bundles while y , x^2 generate the other), but none of these self-covers is regular cyclic (otherwise L^3 would not be a codimension 2 fibrator [Da, Theorem 4.2]).

Remarks. Both S^3 and P^3 are known to be codimension 2 fibrators (cf. [Da, Theorems 2.1 and 6.1]). The techniques developed here show that additional manifolds with S^3 geometric structure share the property; namely, those with fundamental group equal to a dihedral group $D_{2^m \cdot (2k+1)}$ ($m > 1$). We do not know whether the same holds for Lens spaces $L(p, q)$ when $p > 2$. Analysis similar to that of Lemma 8 reveals that many Siefert fibered 3-manifolds with other geometric structures and finite first homology are codimension 2 fibrators.

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DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TENNESSEE, KNOXVILLE, TENNESSEE
37996-1300