PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 113, Number 2, October 1991

# MANIFOLDS WITH FINITE FIRST HOMOLOGY AS CODIMENSION 2 FIBRATORS

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(Communicated by James E. West)

ABSTRACT. Given a map $f: M \to B$ defined on an orientable $(n+2)$-manifold with all point inverses having the homotopy type of a specified closed $n$-manifold $N$, we seek to catalog the manifolds $N$ for which $f$ is always an approximate fibration. Assuming $H_1(N)$ finite, we deduce that the cohomology sheaf of $f$ is locally constant provided $N$ admits no self-map of degree $d > 1$ when $H_1(N)$ has a cyclic subgroup of order $d$. For manifolds $N$ possessing additional features, we achieve the approximate fibration conclusion.

## 1. INTRODUCTION

Codimension 2 fibrators, a notion introduced in [Da], afford quick recognition of certain maps $p: M \to B$ as approximate fibrations. This, in turn, gives rise to reasonably computable structural information, based on a long exact homotopy sequence for approximate fibrations, developed by Coram and Duvall [CD1], analogous to the classical one for fibrations. Known codimension 2 fibrators include all simply connected closed manifolds, closed 2-manifolds of negative Euler characteristic, and $P^n$ (real projective $n$-space); nonfibrators in codimension 2 include all closed manifolds admitting a fixed point free cyclic action having orbit space homotopy equivalent to itself [Da].

A closed, connected manifold $N^n$ is called a *codimension* 2 (*orientable*) *fibrator* if whenever $f: M \to B$ is a proper closed map defined on an (orientable) $(n+2)$-manifold and each $f^{-1}(b)$ is a compact ANR with the homotopy type of $N^n$, $f$ is an approximate fibration (meaning, $f$ satisfies the approximate homotopy lifting property; see [CD1]). In this setting, the upper semicontinuous (henceforth, abbreviated as usc) decomposition $\{f^{-1}(b): b \in B\}$ of $M$ induced by $f$ is called $N^n$*-like.* Instead of the ANR hypothesis, one could just as well require all $f^{-1}(b)$ in an $N^n$-like decomposition to be compact sets shape equivalent to $N^n$.

Received by the editors February 8, 1989 and, in revised form, September 5, 1989.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 55R65, 57N15, 57N65; Secondary 55M25, 57N12, 54B15.

*Key words and phrases.* Approximate fibration, approximate lifting, continuity set, codimension 2 fibrator, geometric structure, Hopfain group, mapping degree.

Research supported in part by a grant from the National Science Foundation.

By the *degree* of a map between closed, connected, orientable $n$-manifolds, we mean more precisely the absolute degree; namely, the nonnegative integer determining the induced homomorphism's effect on $n$th (integral) homology, up to sign. Let $N^n$ be a closed orientable $n$-manifold and $G$ an $N^n$-like decomposition of an orientable $(n+2)$-manifold $M$. Then the decomposition space $B = M/G$ is a 2-manifold [DW]. Further analysis depends on the concept of the *continuity set* $C \subset B$ of the decomposition map $p: M \to B$, which we briefly describe now in terms of degree: each $g \in G$ has a neighborhood $U_g$ equipped with a retraction $R_g: U_g \to g$, and $p(g) \in C$ iff $R_g$ restricts to a degree 1 map $g' \to g$ for all $g' \in G$ sufficiently close to $g$. By [DW] $C$ contains all but a locally finite subset of $B$.

Recall that a group $\Gamma$ is *Hopfian* if every epimorphism $\Gamma \to \Gamma$ is necessarily $1-1$.

In lieu of additional review, we restate two basic facts from [Da] in a form readily applicable here.

**Proposition AF.** *Suppose $N^n$ is a closed orientable n-manifold such that either $\pi_1(N^n)$ is finite or it is Hopfian and $N^n$ is aspherical, and suppose $G$ is an $N^n$-like usc decomposition of an orientable $(n+2)$-manifold $M$. Then the decomposition map $p: M \to B$ restricts to an approximate fibration $p|p^{-1}(C): p^{-1}(C) \to C$ over the continuity set $C$ of $p$.*

We close by outlining the strategy. Under the hypothesis that $H_1(N^n)$ is finite, we seek to prove that the continuity sets $C$ of $p: M \to B$ associated with various $N^n$-like decompositions satisfy $C = B$. The homological finiteness condition alone does not force $N^n$ to be a codimension 2 fibrator— $P^{2k+1} \# P^{2k+1}$ is a counterexample [Da]. Nevertheless, then all $R_g$ restrict to maps $g' \to g$ with nonzero degree, useful data in itself, and the first result in the next section limits the possible degrees $d$ to the orders of cyclic subgroups of $H_1(N^n)$. Of course, $C = B$ when $d = 1$ is the only possibility. Finally, we impose further conditions on $N^n$, like those in the hypothesis of Proposition AF, to detect approximate fibrations.

## 2. Manifolds with finite first homology

**Proposition 1.** *Suppose $N^n$ is a closed, orientable n-manifold whose first (integral) homology group is finite. Suppose $M$ is any orientable $(n+2)$-manifold, $G$ is an $N^n$-like usc decomposition of $M$, $g_0 \in G$, and $R: U \to g_0$ is a retraction defined on a neighborhood $U$ of $g_0$. Then there exists an integer $d \geq 1$ such that*

(1) *$g_0$ has a neighborhood $U' \subset U$ where, for all $g \in G$ in $U'$, $g \neq g_0$, $R|g: g \to g_0$ has absolute degree $d$;*
(2) *$H_1(N^n)$ contains a cyclic subgroup of order $d$.*

*Proof*. At the core of the argument, which mostly involves algebraic diagram-chasing, is the simplification stemming from the relationships $H_{n-1}(N^n) \cong H^1(N^n) \cong 0$ insured by duality and universal coefficients.

Consider the decomposition map $p: M \to B = M/G$. In this setting $B$ is a 2-manifold [DW]. Restricting $M$, we reduce to the situation where $B$ is an open 2-cell containing $b_0 = p(g_0)$ and $p: M \to B$ is an approximate fibration over $B \setminus b_0$.

Application of approximate lifting over $B \setminus b_0$ shows $M$ deformation retracts to $g_0$, with $R: M \to g_0$ being the end of the deformation. Accordingly, $R_*: H_*(M) \to H_*(g_0)$ is an isomorphism. From the exact sequence of the pair $(M, M \setminus g_0)$,

$$0 \cong H_{n+2}(M) \to H_{n+2}(M, M \setminus g_0) \to H_{n+1}(M \setminus g_0) \to H_{n+1}(M) \cong 0,$$

yielding $H_{n+1}(M \setminus g_0) \cong Z$, and

$$0 \cong H^1(g_0) \cong H_{n+1}(M, M \setminus g_0) \to H_n(M \setminus g_0) \to H_n(M) \cong Z.$$

Finiteness of $H_1(N^n)$ is crucial to proving $H_n(M \setminus g_0)$ is the (isomorphic) image of $H_n(g)$ for any $g \in G \setminus \{g_0\}$. Express $M \setminus g_0$ as the union of $G$-saturated open sets $V$, $V'$ whose intersection consists of components $W$, $W'$, where $p(V)$, $p(V')$, $p(W)$, $p(W')$ are all 2-cells and $g \subset W$. Since

$$H_n(V) \cong H_n(W) \cong H_n(g) \cong Z \cong H_n(V') \cong H_n(W'),$$
$$H_{n-1}(W) \cong H_{n-1}(g) \cong 0 \cong H_{n-1}(W'),$$

the Mayer–Vietoris sequence for the triad $(M \setminus g_0, V, V')$ shows

$$(\zeta) \qquad 0 \to Z \cong H_{n+1}(M \setminus g_0) \to Z \oplus Z \xrightarrow{\psi_*} Z \oplus Z \to H_n(M \setminus g_0) \to 0.$$

Focus on the homomorphism $H_n(W) \oplus H_n(W') \to H_n(V) \oplus H_n(V')$ giving rise to the homomorphism $\psi_*: Z \oplus Z \to Z \oplus Z$ in $(\zeta)$, and arrange notation to make $\psi_*(\langle s, t\rangle) = \langle s + t, s \pm t\rangle$. Sequence $(\zeta)$ shows $\ker \psi_* \neq 0$, so $\psi_*(\langle s, t\rangle) = \langle s+t, s+t\rangle$ and, thus, $H_n(M \setminus g_0) \cong (Z \oplus Z)/\operatorname{im}(\psi_*)$ corresponds to $H_n(V) \oplus 0 \cong H_n(g) \oplus 0$.

Now examine the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_n(M \setminus g_0) & \xrightarrow{i_*} & H_n(M) & \xrightarrow{j_*} & H_n(M, M \setminus g_0) \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ & & Z & & Z & & H^2(g_0) \cong \text{Free} \oplus H_1(g_0). \end{array}$$

Up to sign, $i_*$ amounts to multiplication by an integer $d \geq 1$, and $j_*$ injects the cokernel, a cyclic subgroup of order $d$, in $H_1(g_0) \cong H_1(N^n)$. Finally, $R|g: g \to g_0$ has degree $d$, since the induced homomorphism on $n$th homology can be expressed as the composition of isomorphisms with $i_*$.

**Corollary 2.** *Suppose $N^n$ is a closed, orientable, n-manifold such that $H_1(N^n)$ is finite and whenever $H_1(N^n)$ contains a cyclic subgroup of order $d > 1$, then*

*$N^n$ admits no self-map of degree $d$. Suppose $G$ is an $N^n$-like usc decomposition of an orientable $(n+2)$-manifold $M$. Then the (integral) cohomology sheaf of the decomposition map $f\colon M \to B = M/G$ is locally constant.*

*Proof.* Clearly the maps $R|g\colon g_0$ induce isomorphisms on $n$th homology. By Poincaré duality and naturality of cap products, they have the same effect on homology groups at all levels (application of 5.6.16 from [Sp, p. 254] reveals that these $(R|g)^*$ provide epimorphisms and hence isomorphisms between $(n-q)$th homology groups). Duality implies these $(R|g)^*$ are cohomology isomorphisms in all dimensions, from which the corollary follows.

Propositions 1 and AF combine as:

**Theorem 3.** *Suppose $N^n$ is a closed, orientable, n-manifold for which either $\pi_1(N^n)$ is finite or $\pi_1(N^n)$ is Hopfian, $H_1(N^n)$ is finite, and $N^n$ is aspherical. Suppose further that if $H_1(N^n)$ contains a cyclic subgroup of order $d > 1$ then $N^n$ admits no self-map of degree $d$. Then $N^n$ is a codimension 2 orientable fibrator.*

**Corollary 4.** *If $\Gamma$ is a finite group admitting no homomorphisms $\psi\colon \Gamma \to \Gamma$ with index $[\Gamma, \psi(\Gamma)] > 1$ equal to the order of a cyclic subgroup of the abelianization, $\Gamma/\operatorname{comm}(\Gamma)$, then every orientable manifold $N$ with $\pi_1(N) \cong \Gamma$ is an orientable codimension 2 fibrator.*

*Proof.* Given $N$ with fundamental group $\Gamma$, consider any map $f\colon N \to N$ of degree $d > 1$, where $d$ is the order of a cyclic subgroup of $H_1(N) \cong \Gamma/\operatorname{comm}(\Gamma)$. Suppose $f_\#(\Gamma) \neq \Gamma$. Specify a covering space $q\colon N^* \to N$ corresponding to $f_\#(\Gamma)$ and a lift $f^*\colon N \to N^*$ of $f$. Since

$$d = \deg(f) = \deg(q) \cdot \deg(f^*),$$

we see that $[\Gamma, f_\#(\Gamma)] = \deg(q)$ divides $d$, implying the existence of another cyclic subgroup in $H_1(N)$ of order $[\Gamma, f_\#(\Gamma)]$, contrary to hypothesis. Thus, all degree $d$ maps between copies of $N$ encountered in our setting induce $\pi_1$-isomorphisms, in which case we can imitate the proof of [Da, Proposition 2.6] to conclude $N$ is a codimension 2 fibrator.

For examples, such familiar objects as the symmetric group $\mathscr{S}_m$ on $m > 4$ symbols, the nonabelian group of order 21, and the dihedral groups

$$D_{2^m\cdot(2k+1)} = \langle x, y | x^{2^m} = 1 = y^{2k+1}, \ xyx^{-1} = y^{-1}\rangle$$

all satisfy the hypothesis of Corollary 4. Inspecting the third example more closely, we see it abelianizes to the cyclic group of order $2^m$ and any subgroup $H$ of index $2^s$ is cyclic (because $y$ commutes with $x^2$) of order divisible by $2k+1$, but obviously then this dihedral group does not surject to $H$.

Finite abelian groups are not amenable to this analysis.

## 3. Applications to 3-manifolds

We point out another fact closely related to Proposition AF.

**Proposition 5.** *Suppose $N^n$ is a closed orientable n-manifold, $n \in \{3, 4\}$, with $\pi_1(N^n)$ Hopfian and suppose $G$ is an $N^n$-like usc decomposition of an orientable $(n+2)$-manifold $M$. Then the decomposition map $p: M \to B$ restricts to an approximate fibration $p|p^{-1}(C)$ over the continuity set $C$ of $p$.*

*Proof.* Hausmann [Ha, Proposition 1(b)] establishes that, over $C$, the retractions $R_g: U_g \to g$ restrict to homotopy equivalences $g' \to g$ for $g' \in G$ sufficiently close to $g$. In the context at hand this property yields the conclusion [CD2, Corollary 3.4].

Conceivably the class of manifolds $N^3$ described below is all-inclusive. Under this hypothesis Hempel [He2, Corollary 1.2] proves $\pi_1(N^3)$ is Hopfian (also see [He1] for the revelant definition).

**Corollary 6.** *Suppose $N^3$ is a closed, orientable 3-manifold whose prime factors are either virtually Haken or else have finite or cyclic fundamental groups, and suppose $G$ is an $N^3$-like usc decomposition of an orientable 5-manifold $M$. Then the decomposition map $p: M \to B$ restricts to an approximate fibration over its continuity set.*

**Corollary 7.** *Every closed hyperbolic 3-manifold with finite first homology is a codimension 2 fibrator.*

*Proof.* According to Gromov [Gr, p. 147], there is no degree $d > 1$ self-map on a closed hyperbolic manifold. (Lemma 4.1 of [He2] certifies the Hopfian part needed in Proposition 5.)

The conclusion of Corollary 7 might be valid without the $H_1$-finiteness condition.

Let $L^3$ denote the unique closed 3-manifold having Euclidean structure and finite first homology (in other words, $L^3$ does not fiber over $S^1$). It is the orientable Seifert bundle over $P^2$ with Euclidean structure appearing in Scott's list [Sc, p. 446]. Standard calculations from the associated Seifert bundle data yield $H_1(L^3) \cong Z_4 \oplus Z_4$.

**Lemma 8.** *There is no map $L^3 \to L^3$ of degree 2 or 4.*

*Proof.* Suppose $f: L^3 \to L^3$ is a map of degree 2 or 4. Keep in mind that $f_\#(\pi_1(L^3)) \neq \pi_1(L^3)$ for otherwise $f_\#$ would be an isomorphism and, as a self-map on a $K(\pi, 1)$ space, $f$ would be a homotopy equivalence and, therefore, have degree 1. As in the proof of Corollary 4, take a (non-trivial) covering $q: N^* \to L^3$ corresponding to $f_\#(\pi_1(L^3))$ and lift $f$ to $f^*: L^3 \to N^*$. Significant features are: $f^*$ induces an epimorphism at the $H_1$ level (of course, on $\pi_1$ as well), and $q$ has degree 2 or 4.

We can regard $L^3$ as the result of identifying two copies $T$, $T'$ of the orientable twisted $I$-bundle over a Klein bottle via the homeomorphism $\theta$ on the boundary torus interchanging the $S^1$-factors, in some preferred parameterization. (Scott [Sc, p. 448] relates that closed Euclidean 3-manifolds arise either as torus or Klein bottles bundles over $S^1$ or as the union of two twisted $I$-bundles, and the finiteness of $H_1(L^3)$ precludes $L^3$ from being among the former.) Thus, $N^*$ is a union of finite-sheeted covers of $T$, $T'$, which necessarily are either $I$-bundles over the torus or homeormorphic lifts of $T$ and $T'$ themselves.

Suppose $q$ is 2-sheeted. Then $N^*$ could be a torus bundle over $S^1$ (with monodromy $\theta$); this occurs iff both twisted $I$-bundles lift to $S^1 \times S^1 \times I$. Otherwise $N^*$ is another union of twisted $I$-bundles over a Klein bottle. The latter occurs when (precisely) one of $T$, $T'$ is covered by two copies of itself and $N^*$ coincides with their union plus a boundary collar); a check of the various lifts shows the attaching homeomorphism to be the identity. Neither case can arise in the present context, since each requires $H_1(N^*)$ to be infinite.

Moreover, $q$ cannot be 4-sheeted either, for it follows easily that $q$ would factor through a 2-sheeted cover, again implying $H_1(N^*)$ is infinite.

**Corollary 9.** *$L^3$ is an orientable codimension 2 fibrator.*

Among closed 3-manifolds having Euclidean geometric structure, $L^3$ is the solitary example satisfying the conclusion of Corollary 9. Like all Euclidean manifolds, $L^3$ covers itself—in fact, it does so $k$ times for every odd integer $k$ (e.g., the cover corresponding to the subgroup generated by $x^k$, $y^2$ in the presentation for $\pi_1(L^3)$ as

$$\langle x, y | xy^2 = y^{-2}x, yx^2 = x^{-2}y \rangle,$$

where $x$, $y^2$ generate the contribution of one of these twisted $I$-bundles while $y$, $x^2$ generate the other), but none of these self-covers is regular cyclic (otherwise $L^3$ would not be a codimension 2 fibrator [Da, Theorem 4.2]).

*Remarks.* Both $S^3$ and $P^3$ are known to be codimension 2 fibrators (cf. [Da, Theorems 2.1 and 6.1]). The techniques developed here show that additional manifolds with $S^3$ geometric structure share the property; namely, those with fundamental group equal to a dihedral group $D_{2^m \cdot (2k+1)}$ $(m > 1)$. We do not know whether the same holds for Lens spaces $L(p, q)$ when $p > 2$. Analysis similar to that of Lemma 8 reveals that many Siefert fibered 3-manifolds with other geometric structures and finite first homology are codimension 2 fibrators.

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