HOMOTOPY-COMMUTATIVE $H$-SPACES

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(Communicated by Frederick R. Cohen)

Abstract. Let $X$ be an $H$-space with $H^*(X; Z_2) \simeq Z_2[x_1, \ldots, x_d] \otimes \Lambda(y_1, \ldots, y_d)$, where $\deg x_i = 4$ and $y_i = Sq^1 x_i$. In this article we prove that $X$ cannot be homotopy-commutative. Combining this result with a theorem of Michael Slack results in the following theorem: Let $X$ be a homotopy-commutative $H$-space with mod 2 cohomology finitely generated as an algebra. Then $H^*(X; Z_2)$ is isomorphic as an algebra over $A(2)$ to the mod 2 cohomology of a torus producted with a finite number of $CP(\infty)$s and $K(Z_2, 1)$s.

0. Introduction

In this article we prove the following theorem:

Theorem A. Let $X$ be an $H$-space with

$$H^*(X; Z_2) = Z_2[x_1, \ldots, x_d] \otimes \Lambda(y_1, \ldots, y_d),$$

where $\deg x_i = 4$ and $y_i = Sq^1 x_i$. Then $X$ cannot be homotopy-commutative.

The significance of Theorem A lies in its relationship to the following theorem, due to Michael Slack:

Theorem (Slack). Let $X$ be a homotopy-commutative $H$-space with mod 2 cohomology finitely generated as an algebra. Then

1. All even-degree generators have infinite height and are in degrees two and four.
2. All odd-degree generators lie in degrees one and five. The one-dimensional generators have infinite height and the five-dimensional generators are exterior.
3. $Sq^1 : QH^4(X; Z_2) \rightarrow QH^5(X; Z_2)$ is an isomorphism.

Received by the editors February 27, 1990.
1980 Mathematics Subject Classification (1985 Revision). Primary 55P45, 55S40.
Key words and phrases. Homotopy-commutative $H$-space, cohomology operation, Steenrod algebra.

The first author's work was supported by a grant from the National Science Foundation.
Combining Theorem A with Slack's theorem results in the following:

**Theorem B.** Let $X$ be a homotopy-commutative $H$-space with mod 2 cohomology finitely generated as an algebra. Then $H^*(X; \mathbb{Z}_2)$ is isomorphic as an algebra over $A(2)$ to the mod 2 cohomology of a torus producted with a finite number of $CP(\infty)$s and $K(\mathbb{Z}_2', 1)$s.

This theorem extends work of Hubbuck and Lin [3, 4] which proves any finite homotopy-commutative $H$-space has the homotopy type of a torus. Here we treat the larger class of $H$-spaces whose cohomology is finitely generated as an algebra.

Theorem A also has the following immediate corollary:

**Corollary C.** The 3-connective cover of $S^3$, $\tilde{S}^3$, is not mod 2 homotopy-commutative, hence neither is any product of $S^3$'s homotopy-commutative.

**Remarks.** (1) Corollary C can also be proved by using results in [8] and verifying that the Samelson product $\langle \alpha, \alpha \rangle \neq 0$, where $\alpha$ is a generator of $\pi_4(\tilde{S}^3)$. This approach was suggested to the authors by Chuck McGibbon. (The referee has observed that Corollary C can also be obtained quickly from results of Zabrodsky.)

(2) Our original proof of Theorem A rested upon an analysis of a partial Postnikov tower for $X$. The proof we give here is shorter and is based on a factorization of $Sq^8 Sq^4$ (Theorem 1.1 below) that may well be of independent interest.

(3) In the rest of this paper, cohomology will be taken to have coefficients in $\mathbb{Z}_2$ unless otherwise specified.

### 1. Factorization of $Sq^8 Sq^4$

**Theorem 1.1.** Let $x \in H^8(X; \mathbb{Z})$ be such that $\rho(x)$ lies in the kernel of $Sq^2$, where $\rho: H^*(X; \mathbb{Z}) \to H^*(X; \mathbb{Z}_2)$ is reduction mod 2. Then there is a formula

$$Sq^8 Sq^4 x = Sq^4 \phi_{0,3}(x) + (Sq^9 + Sq^{7,2})\varphi_{1,1}(x) + x(Sq^4 x),$$

where the secondary operations $\phi_{0,3}$ and $\varphi_{1,1}$ are defined by the following relations:

$$\phi_{0,3}: Sq^9 = (Sq^7 + Sq^{4,2,1}) Sq^2 \text{ on an integral class}$$

$$\varphi_{1,1}: Sq^2 Sq^2 = 0 \text{ on an integral class}.$$
Proof. Consider the following two-stage Postnikov system.

\[ K(\mathbb{Z}_2, 9) \xrightarrow{j} E \xrightarrow{p} K(\mathbb{Z}, 8) \xrightarrow{g} K(\mathbb{Z}_2, 10) \]

in which \( g^*(t_{10}) = \text{Sq}^2 t_8 \). Then there are elements \( \tilde{v}_{0,3} \in H^{16}(E) \) and \( v_{1,1} \in H^{11}(E) \) such that

\[ j^*(\tilde{v}_{0,3}) = (\text{Sq}^7 + \text{Sq}^{4,2,1}) t_9 \]

and

\[ j^*(v_{1,1}) = \text{Sq}^2 t_9. \]

Further, \( \Delta(\tilde{v}_{0,3}) = p^*(t_8) \otimes p^*(t_8) \), where \( \Delta \) denotes the reduced diagonal. We compute

\[ j^*((\text{Sq}^9 + \text{Sq}^{7,2}) v_{1,1} + \text{Sq}^4 \tilde{v}_{0,3}) = 0 \]

and

\[ \Delta((\text{Sq}^9 + \text{Sq}^{7,2}) v_{1,1} + \text{Sq}^4 \tilde{v}_{0,3}) = \Delta(p^*(t_8) \text{Sq}^4 p^*(t_8)). \]

Hence

\[ z = (\text{Sq}^9 + \text{Sq}^{7,2}) v_{1,1} + \text{Sq}^4 \tilde{v}_{0,3} - p^*(t_8) \text{Sq}^4 p^*(t_8) \]

is an element of \( PH^{20}(E) \cap \ker(j^*) \). So \( z \in p^*PH^{20}(K(\mathbb{Z}, 8)) \), and thus by [2] \( z = c \text{Sq}^{8,4} p^*(t_8) \), for some \( c \in \mathbb{Z}_2 \).

To prove the theorem, we must show that \( c = 1 \). To do this, we loop the diagram four times, obtaining

\[ K(\mathbb{Z}_2, 5) \xrightarrow{j} \Omega^4 E \xrightarrow{\rho} K(\mathbb{Z}, 4) \xrightarrow{g} K(\mathbb{Z}_2, 6) \]

Since \( \text{Sq}^5 t_4 = \text{Sq}^{2,1} \text{Sq}^2 t_4 \), there exists \( \tilde{v}_{0,2} \in H^8(\Omega^4 E) \) such that \( j^*(\tilde{v}_{0,2}) = \text{Sq}^{2,1} t_5 \) and \( \Delta(\tilde{v}_{0,2}) = \tilde{p}^*(t_4) \otimes \tilde{p}^*(t_4) \). So \( \Delta(\text{Sq}^4 \tilde{v}_{0,2}) = \Delta(\tilde{p}^*(t_4)^3) \). Thus

\[ \text{Sq}^4 \tilde{v}_{0,2} + \tilde{p}^*(t_4)^3 + (\sigma^*)^4(\tilde{v}_{0,3}) \in PH^{12}(\Omega^4 E) \cap \ker(j^*) \]

\[ \subset \tilde{p}^*PH^{12}(K(\mathbb{Z}, 4)) = 0. \]
We have
\[ \text{cp}^*(t_4) = c \text{Sq}^4 \text{p}^*(t_4) = \text{Sq}^9 + \text{Sq}^{7,2} v_{1,1} + \text{Sq}^4 \vartheta_{0,3} \]
\[ = \text{Sq}^{7,2} (\sigma^*) v_{1,1} + \text{Sq}^4 \vartheta_{0,3} \]
\[ = \text{Sq}^{7,2} (\sigma^*) v_{1,1} + \text{Sq}^4 \text{Sq}^4 \vartheta_{0,2} + \text{Sq}^4 \text{p}^*(t_4)^3. \]

Let \( f: K(Z, 2) \to K(Z, 4) \) be defined by \( f^*(t_4) = t_2^2 \). Let \( f: K(Z, 2) \to \Omega^4 \) be a lifting of \( f \). Then
\[ c l_2^8 = \tilde{f}^* (\text{Sq}^{7,2} (\sigma^*) v_{1,1} + \text{Sq}^4 \text{Sq}^4 \vartheta_{0,2} + \text{Sq}^4 \text{p}^*(t_4)^3) \]
\[ = 0 + \lambda \text{Sq}^4 \text{t}_2^4 + \text{Sq}^4 \text{t}_2^6, \quad \lambda = 0 \text{ or } 1 \]
\[ = l_2^8. \]

So \( c = 1. \) □

2. Relations in \( P_2X \)

Henceforth, we will assume that \( X \) is an \( H \)-space with mod 2 cohomology,
\[ H^*(X) = Z_2 [x_1, \ldots, x_d] \otimes \Lambda (y_1, \ldots, y_d) \]
as in the statement of Theorem A. For degree reasons, \( H^*(X) \) is a primitively
generated Hopf algebra.

According to [1], there is an exact triangle
\[ H^*(P_2X) \xrightarrow{i} H^*(X) \xrightarrow{\sqrt{\Delta}} \]
\[ \Delta H^*(X) \otimes H^*(X) \]
in which \( i \) has degree \(-1\), \( \lambda \) has degree 2, and \( \Delta \) has degree 0.

Therefore there exist unique elements \( u_i \in H^5(P_2X) \) and \( v_i \in H^6(P_2X) \) such
that \( i(u_i) = x_i \), \( i(v_i) = y_i \), and \( \text{Sq}^1 u_i = v_i \). A simple calculation following [1]
shows that
\[ Z_2 [u_1, \ldots, u_d, v_1, \ldots, v_d, \text{Sq}^4 u_1, \ldots, \text{Sq}^4 u_d] \subset H^*(P_2X). \]

Note that \( \text{Sq}^2 H^6(P_2X) = 0 \). Using the Adem relation \( \text{Sq}^2 \text{Sq}^3 = \text{Sq}^5 + \text{Sq}^4 \text{Sq}^1 \)
and the fact that \( \text{Sq}^3 u_i = 0 \), we obtain
\[ \text{Sq}^4 v_i = \text{Sq}^4 \text{Sq}^4 u_i = \text{Sq}^5 u_i = \text{Sq}^1 \text{Sq}^4 u_i. \]

Hence, if \( \rho: H^*(P_2X; Z) \to H^*(P_2X; Z_2) \) is mod 2 reduction, we have
\[ \text{Sq}^4 H^6(P_2X; Z_2) \subset \text{Im}(\rho). \]
Consider the following commutative diagram of infinite loop maps

\[
\begin{array}{ccc}
K(Z_2; 7, 9) & \rightarrow & K(Z_2; 7, 9) \\
\downarrow j & & \downarrow j \\
E & \xrightarrow{h} & E \\
\downarrow p & & \downarrow \bar{p} \\
K(Z_2, 5) \times K(Z, 10) & \xrightarrow{\delta} & K(Z, 6, 10) \\
\downarrow w & & \downarrow \bar{w} \\
K(Z_2; 8, 10) & \rightarrow & K(Z_2; 8, 10)
\end{array}
\]  

(2.4)

in which

\[
\begin{align*}
\overline{w}^*(t_8) &= \text{Sq}^2 t_6; \\
\overline{w}^*(t_{10}) &= \text{Sq}^4 t_6 - p(t_{10}); \\
w^*(t_8) &= \text{Sq}^2 \text{Sq}^1 t_5; \\
w^*(t_{10}) &= \text{Sq}^4 \text{Sq}^1 t_5 - p(t_{10}); \\
\end{align*}
\]

and

\[
\delta^*(t_{10}) = t_{10}; \quad \delta^*(t_6) = \text{Sq}^1 t_5.
\]

There exist elements \(v_{1, 1} \in PH^9(E)\), \(v_{0, 2} \in PH^{10}(E)\) such that \(\overline{\delta}^*(v_{1, 1}) = \text{Sq}^2 t_7\) and \(\overline{\delta}^*(v_{0, 2}) = \text{Sq}^2 \text{Sq}^1 t_7 + \text{Sq}^1 t_9\).

**Lemma 2.1.** If \(v_{1, 1} = h^*(v_{1, 1})\) and \(v_{0, 2} = h^*(v_{0, 2})\), then \(\text{Sq}^2 v_{1, 1} = \text{Sq}^1 v_{0, 2}\).

**Proof.** Note that

\[
\text{Sq}^2 \overline{v}_{1, 1} + \text{Sq}^1 \overline{v}_{0, 2} \in \ker \overline{\delta}^* = \overline{\delta}^*(PH^{11}(K(Z; 6, 10))) = 0.
\]

Therefore \(\text{Sq}^2 \overline{v}_{1, 1} = \text{Sq}^1 \overline{v}_{0, 2}\). Applying \(h^*\) gives the lemma. \(\Box\)

Define \(\varphi_{1, 1}\) and \(\varphi_{0, 2}\) to be the secondary operations defined by the elements \(v_{1, 1}\) and \(v_{0, 2}\), respectively. We note that if \(y \in H^5(P_2 X)\), then \(\varphi_{1, 1}\) and \(\varphi_{0, 2}\) are defined on \(\text{Sq}^1 y\) with zero indeterminacy.

We have the following relations:

1. \(\varphi_{1, 1} \text{Sq}^1 H^5(P_2 X) \subset \text{Sq}^4 H^5(P_2 X)\), since \(H^9(P_2 X) = \text{Sq}^4(P_2 X)\).
2. \(\text{Sq}^2 \varphi_{1, 1} \text{Sq}^1 H^5(P_2 X) = \text{Sq}^1 \varphi_{0, 2} \text{Sq}^1 H^5(P_2 X)\), by Lemma 2.1.

**Proposition 2.2.** \(\varphi_{0, 2} \text{Sq}^1 H^5(P_2 X) \subset \text{Sq}^5 H^5(P_2 X)\).
Proof. Let \( y \) be in \( H^5(P_2X) \). Then
\[
\begin{align*}
\text{Sq}^1 \varphi_{0,2} \text{Sq}^1 y &= \text{Sq}^2 \varphi_{1,1} \text{Sq}^1 y \
&= \text{Sq}^2 \text{Sq}^4 y' \quad \text{by Lemma 2.1} \\
&= \text{Sq}^5 \text{Sq}^1 y' \\
&= \text{Sq}^1 \text{Sq}^4 \text{Sq}^1 y' \\
&= \text{Sq}^1 \text{Sq}^4 \text{Sq}^4 y' \\
&= 0.
\end{align*}
\]
But \( H^{10}(P_2X) \cap \ker \text{Sq}^1 \subset \text{Sq}^5 H^5(P_2X) \). \( \square \)

In the proofs of the next two theorems we will be using properties of the \( c \)-invariant and the \( H \)-deviation. We refer the reader to the appendix of [5] for a summary of general properties of these operations.

**Theorem 2.3.** \( \varphi_{1,1} \text{Sq}^1 H^5(P_2X) = 0 \).

**Proof.** We now construct a Postnikov system which reflects the following relations for \( u \in H^5(P_2X) \):

1. \( \varphi_{1,1} \text{Sq}^1 u = \text{Sq}^4 u' \),
2. \( \varphi_{0,2} \text{Sq}^1 u = \text{Sq}^3 u'' \).

Consider the diagram
\[
\begin{align*}
K(Z_2; 8, 9) &\quad \downarrow \\
E_1 &\quad \downarrow \\
E &\xrightarrow{w} K(Z_2; 9, 10) \\
K(Z_2; 5, 5, 5) \times K(Z; 10, 10) &\quad \xrightarrow{w_0} K(Z_2; 8, 10, 10)
\end{align*}
\]

in which
\[
\begin{align*}
w_0^{*}(t_8) &= \text{Sq}^{2,1} t_5, \\
w_0^{*}(t_{10}) &= \text{Sq}^{4,1} t_5 - t_{10}, \\
w_0^{*}(t'_{10}) &= \text{Sq}^{4,1} t'_5 - t'_{10},
\end{align*}
\]
and
\[
\begin{align*}
w^{*}(t_g) &= \varphi_{1,1} \text{Sq}^1 t_5 - \text{Sq}^4 t'_5, \\
w^{*}(t_{10}) &= \varphi_{0,2} \text{Sq}^1 t_5 - \text{Sq}^5 t''_5.
\end{align*}
\]

We shall retain the notation \( t_5 \) and \( t'_5 \) to denote the images of these elements in \( E \) and \( E_1 \).
In $H^*(E_1)$ we have
\[
\begin{align*}
\text{Sq}^6 i_5' &= (\text{Sq}^2 \text{Sq}^4 + \text{Sq}^5 \text{Sq}^1) i_5' \\
&= \text{Sq}^2 \text{Sq}^4 i_5' + \text{Sq}^1 (\text{Sq}^4 \text{Sq}^1 i_5') \\
&= \text{Sq}^2 \phi_{1,1} \text{Sq}^1 i_5' + \text{Sq}^1 i_{10}' \\
&= \text{Sq}^1 \phi_{0,2} \text{Sq}^1 i_5 \\
&= \text{Sq}^1 \text{Sq}^5 i_5''
\end{align*}
\]

This Postnikov system can be delooped. It follows that there is an element $v \in H^{10}(E_1)$ with $j^*_1(v) = \text{Sq}^2 i_8 + \text{Sq}^1 i_9'$ and $\overline{\Delta} v = i_5' \otimes i_5'$. Therefore $c(\sigma^* v) = \sigma^*(i_5') \otimes \sigma^*(i_5')$. Since $P_2X$ lifts to $E_1$, there is an $H$-lifting $f_1$ of $X$ to $\Omega E_1$. Checking degrees, one sees that $X$ lifts to $\Omega E_1$ by a $c$-map. This implies that
\[
c(f_1^* \sigma^* v) = x' \otimes x' \neq 0, \quad \text{where } u' = i(x')
\]
Therefore $f_1^* \sigma^* v$ is a nonzero primitive in $H^9(X)$. But $PH^9(X) = 0$. □

3. Proof of Theorem A

We can now prove Theorem A.

Proof. By (2.2), if $u \in H^5(P_2X)$, then $(\text{Sq}^4 u)^2 \neq 0$. We have
\[
(\text{Sq}^4 u)^2 = \text{Sq}^9,4 u
\]
\[
= \text{Sq}^8,4,1 u \quad \text{since } \text{Sq}^2 u = 0
\]
\[
= \text{Sq}^4 \phi_{0,3} \text{Sq}^1 u + (\text{Sq}^9 + \text{Sq}^7,2) \phi_{1,1} \text{Sq}^1 u \quad \text{by Theorem 1.1}
\]
\[
= \text{Sq}^4 \phi_{0,3} \text{Sq}^1 u \quad \text{by Theorem 2.3}.
\]

Consider the following Postnikov system:
\[
\begin{array}{c}
\text{K}(\mathbb{Z}_2; 8, 13) \\
\downarrow j_i
\end{array}
\begin{array}{c}
\text{E}_1 \\
\downarrow p_i
\end{array}
\begin{array}{c}
\text{E} \\
\downarrow
\end{array}
\begin{array}{c}
\text{K}(\mathbb{Z}_2; 9, 14) \\
\downarrow w_i
\end{array}
\begin{array}{c}
\text{K}(\mathbb{Z}_2, 5) \\
\downarrow w_0
\end{array}
\begin{array}{c}
\text{K}(\mathbb{Z}_2; 7, 8)
\end{array}
\]

in which
\[
\begin{align*}
w_0^*(i_7) &= \text{Sq}^2 i_5; \quad w_0^*(i_8) = \text{Sq}^2,1 i_5; \\
w_1^*(i_9) &= \phi_{1,1} \text{Sq}^1 i_5; \quad w_1^*(i_{14}) = \phi_{0,3} \text{Sq}^1 i_5.
\end{align*}
\]
In $H^*(E_1)$,
\[
(Sq^4 i_5)^2 = Sq^{8,5} i_5 = Sq^{8,4,1} i_5 = Sq^4 \phi_{0,3} Sq^1 i_5 + (Sq^9 + Sq^{7,2}) \phi_{1,1} Sq^1 i_5 = 0.
\]

Therefore in $H^{16}(\Omega E_1)$ there exists an element $v$ with
\[
(\Omega j_1)^*(v) = Sq^4 i_{12} + (Sq^9 + Sq^{7,2}) i_7,
\]
\[
\Delta v = i_4^2 \otimes i_4^2.
\]

We have the following commutative diagram
\[
\begin{array}{ccc}
K(Z_2; 7, 12) & \xrightarrow{f_1} & \Omega E_1 \\
\downarrow & & \downarrow \Omega w_1 \\
\Omega E & \xrightarrow{\omega E} & K(Z_2; 8, 13) \\
\downarrow f_0 & & \downarrow \Omega w_0 \\
X & \xrightarrow{f} & K(Z_2; 4) & \xrightarrow{\omega w_0} & K(Z_2; 6, 7)
\end{array}
\]

Here $f$ is an $H$-map, since the relations defined by $w_0$ hold in $P_2X$. Further, $c(f)$ factors through $K(Z_2; 4, 5)$, so in fact $f$ is a $c$-map. It follows
\[
(1 - T^*)[D_{j_1}] = \Omega w_1 \circ c(f) \simeq \ast. \quad \text{So } [D_{j_1}] \in H^{12}(X \wedge X) \cap \ker(1 - T^*).
\]

Any terms of the form $x_i \otimes x_j^2 + x_j^2 \otimes x_i$ may be eliminated by changing $f_1$ by $x_i x_j^2$. It follows that
\[
[D_{j_1}] = \sum_{j \neq k} b_{ijk}(x_i \otimes x_j x_k + x_j x_k \otimes x_i), \quad b_{ijk} \in Z_2.
\]

Therefore
\[
\Delta f_1^*(v) = x^2 \otimes x^2 + Sq^4 \Sigma b_{ijk}(x_i \otimes x_j x_k + x_j x_k \otimes x_i) \neq 0.
\]

This implies that if $\langle t, x^2 \rangle \neq 0$, then $t^2 \neq 0$ for $t \in H_*^*(X)$. But all squares are zero in $H_*^*(X)$, since $H_*^*(X)$ is primitively generated. We conclude that $X$ is not homotopy-commutative. $\square$

References


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