QUOTIENTS OF SOLUTIONS
OF LINEAR ALGEBRAIC DIFFERENTIAL EQUATIONS

LEE A. RUBEL

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Abstract. An example is given of two entire functions \( f, g \) that satisfy linear
differential equations with polynomial coefficients, whose quotient \( h = f/g \) is
entire, yet \( h \) satisfies no such differential equation.

On the same day, Alfred van der Poorten and Alexander Eremenko independently asked me the same question that stems from the theorem of Ritt that
if \( f \) and \( g \) are two simple exponential polynomials whose quotient \( h \) is entire, then \( h \) must also be a simple exponential polynomial, which means that
\( h(x) = \sum_{k=1}^{n} a_k e^{\lambda_k x} \), where \( a_k \) and \( \lambda_k \) are complex constants. They asked me
whether the result remains true if "simple exponential polynomial" is replaced
throughout by "entire solution of a linear differential equation with polynomial
coefficients." Our theorem shows that the answer is no, using a "cousin" of a
function shown to me by Walter Hayman for another purpose.

Theorem. Let

\[
\begin{align*}
(1) \quad f(x) &= e^{2\pi i x^2} - 1, \\
(2) \quad g(x) &= e^{2\pi i x} - 1.
\end{align*}
\]

Then \( h(x) = f(x)/g(x) \) is entire, but does not satisfy any linear differential
equation with polynomial coefficients, even though \( f \) and \( g \) do.

Remark. The equations that \( f \) and \( g \) satisfy can be chosen to be monic, a
condition that Eremenko asked for, in addition. This is trivial for \( g \). As for
\( f \), observe that

\[
\begin{align*}
(3) \quad f'(x) &= (4\pi i x) e^{2\pi i x^2}, \\
(4) \quad f''(x) &= (-8\pi^2 x^2 + 4\pi i) e^{2\pi i x^2},
\end{align*}
\]

\[
\begin{align*}
(5) \quad f'''(x) &= -(16\pi^3 i x^3 + 24\pi^2 x) e^{2\pi i x^2}.
\end{align*}
\]
Now it is evident that
\[ f'''(x) + axf''(x) + bf'(x) = 0 \]
for suitable constants \( a \) and \( b \).

Proof of the theorem. First, \( h \) is entire because the denominator vanishes exactly at the integers, and the numerator vanishes at each integer. The idea behind the rest of the proof is that if \( h(x) \) were to satisfy a linear differential equation with polynomial coefficients, then so would
\[ \tilde{h}(x) = (e^{2iex^2} - 1)/(e^{2iex} - 1), \]
which is obtained by replacing \( \pi \) in \( h \) by \( e \). But \( \tilde{h}(x) \) has infinitely many poles, since most of the zeros of the denominator are not cancelled by zeros of the numerator. But a solution of a linear differential equation with polynomial coefficients can have singularities in the finite complex plane only at the zeros of the leading coefficient.

Now for the details. Suppose
\[ p_n(x)h^{(n)}(x) + p_{n-1}(x)h^{(n-1)}(x) + \cdots + p_0(x)h(x) = 0, \]
say as formal power series, where the \( p_j \) are polynomials. Let \( \varphi \) be an automorphism of \( \mathbb{C} \), with \( \varphi(\pi) = e \), and \( \varphi(i) = i \), so that \( \varphi \) fixes \( \mathbb{Q}(i) \), the Gaussian rationals. Note that the \( h^{(j)}(x) \) are rational functions of \( \pi, x, e^{2\pi ix}, e^{2\pi ix^2} \) with coefficients in \( \mathbb{Q}(i) \). Extend the action of \( \varphi \) to \( \mathbb{C}((x)) \), the field generated by the formal power series in \( x \), via
\[ \varphi \left( \sum a_n x^n \right) = \sum \varphi(a_n) x^n. \]
It is easy to see that \( \varphi(x) = x \), \( \varphi(\pi) = e \), \( \varphi(e^{2\pi ix}) = e^{2\pi ix} \), \( \varphi(e^{2\pi ix^2}) = e^{2\pi ix^2} \), and that \( \varphi \) commutes with differentiation. Therefore, applying \( \varphi \) to (8), we would get
\[ \tilde{p}_n(x)\tilde{h}^{(n)}(x) + \tilde{p}_{n-1}(x)\tilde{h}^{(n-1)}(x) + \cdots + \tilde{p}_0(x)\tilde{h}(x) = 0, \]
where
\[ \tilde{p}_k(x) = \varphi(p_k(x)), \quad k = 0, 1, \ldots, n. \]

As explained above, (9) is impossible. Hence (8) is impossible and the theorem is proved.

References


Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, Illinois 61801