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ENTROPY AND COMPLETELY INTEGRABLE HAMILTONIAN SYSTEMS

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ABSTRACT. Let H be a Hamiltonian on a four-dimensional symplectic manifold. Suppose the system is completely integrable and on some nonsingular compact level surface Q the integral is such that the connected components of the set of critical points form submanifolds. Then we prove that the topological entropy of the system restricted to Q is zero. As a corollary we deduce the nonexistence of completely integrable geodesic flows by means of integrals as described above for compact surfaces with negative Euler characteristic.

1. INTRODUCTION

Let M^4 be a four-dimensional symplectic manifold and H a Hamiltonian on M. Denote by $\operatorname{sgrad} H$ the symplectic gradient of H. Let Q be a nonsingular compact level surface of H. Suppose the system is completely integrable; that is, suppose there exists an additional function on M which is independent of H (almost everywhere) and is in involution with H (such a function is called an integral). Restricting this integral to Q gives a smooth function f. Denote by h_{top} the topological entropy of the Hamiltonian flow restricted to Q. In this article we want to prove the following:

Theorem 1. *Let M^4 be a smooth symplectic manifold and let* $\operatorname{sgrad} H$ *be a Hamiltonian field on M^4. Suppose that the system is completely integrable and that on some nonsingular compact level surface Q the integral f verifies either of the following conditions*:

(a) *f is real analytic.*
(b) *The connected components of the set of critical points of f form submanifolds.*

Then $h_{\text{top}} = 0$.

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Corollary 1. *Let N^2 be a compact connected surface. Assume N^2 supports a geodesic flow that is completely integrable by means of an integral like the one in Theorem 1. Then $\chi(N^2) \geq 0$.*

Remark 1. Corollary 1 was proved by Kozlov [K] in the case of an analytic integral by completely different methods. If we assume condition (b), the integral could even be of class C^1.

Observe that the class of functions considered in (b) includes the Bott integrals studied by Fomenko in [F].

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2. Proofs

We first state a result of Katok that we will use:

Theorem 2 ([Ka, Corollary (4.3)]). *If g is a $C^{1+\alpha}$ $(\alpha > 0)$ diffeomorphism of a compact two-dimensional manifold and $h_{\text{top}}(g) > 0$, then g has a hyperbolic periodic point with a transversal homoclinic point, and consequently there exists a g-invariant hyperbolic set Λ such that the restriction of g to Λ is topologically conjugate to a subshift of finite type.*

We note that Theorem 2 extends to flows without singularities on 3-manifolds. Theorem 1 follows from Theorem 2 and the following lemma.

Lemma 1. *Under the hypothesis of Theorem 1 there are no transversal homoclinic orbits.*

Proof. If the function is real analytic, Lemma 1 was proved by Moser in [M]. Therefore assume that condition (b) is verified. Denote by $\operatorname{Crit}(f)$ the set of critical points of f. Since f is an integral, the flow of $\operatorname{sgrad} H$ leaves $\operatorname{Crit}(f)$ invariant. Condition (b) says that $\operatorname{Crit}(f)$ is a disjoint union of circles and compact connected surfaces. These surfaces are tori and Klein bottles because $\operatorname{sgrad} H$ is never zero.

Suppose now that there is a transversal homoclinic orbit. Then we have the analogue for flows of the hyperbolic set Λ in Theorem 2. We will also call if Λ (for the properties of shifts and suspended horseshoes, we refer to [S]). We claim that there exists a surface X^2 in $\operatorname{Crit}(f)$ such that $\Lambda \subset X^2$. To prove this, observe first that since f is an integral it follows that if γ is a hyperbolic closed orbit, then $\gamma \subset \operatorname{Crit}(f)$. Otherwise the symplectic gradient of f would generate a nonzero eigenvector with eigenvalue 1 for the Poincaré map of γ. This idea can be traced back to Poincaré (see [K] for details). But the hyperbolic closed orbits in Λ are dense, and $\operatorname{Crit}(f)$ is a closed set; therefore $\Lambda \subset \operatorname{Crit}(f)$. Moreover, since the flow on Λ is transitive (i.e., there is a dense orbit), we deduce the claim.

We now consider the flow of $\operatorname{sgrad} H$ restricted to X^2. Since $\operatorname{sgrad} H$ is never zero, for any closed orbit γ, $X^2 - \gamma$ is a cylinder or a Möbius band. By a Poincaré-Bendixson argument (see [PM, p. 34, Exercises 4 and 5]) we deduce

that $\operatorname{sgrad} H$ has no nontrivial recurrent orbits, i.e., if $\omega(\gamma)$ denotes the limit set of the orbit γ and $\gamma \subset \omega(\gamma)$, then $\omega(\gamma)$ is a closed orbit. But this is absurd because dense orbits in Λ have nontrivial recurrence. The lemma is proved.

Remark 2. Note that the proof of Lemma 1 still works if we allow the surfaces to have boundaries. We also note that Moser in [M] proves that $\Lambda \subset \operatorname{Crit}(f)$ with a different argument that needs only that f be of class C^1.

Proof of the corollary. In [D], Dinaburg proved that if $\pi_1(N^2)$ has exponential growth, then $h_{\text{top}} > 0$. Hence, from Theorem 1, we get that $\pi_1(N^2)$ cannot have exponential growth, and therefore $\chi(N^2) \geq 0$.

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