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Z_p ACTIONS ON SPACES OF COHOMOLOGY TYPE $(a, 0)$

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ABSTRACT. A space X that has the cohomology of the one-point union $P^2(n)VS^{3n}$ or $S^nVS^{2n}VS^{3n}$ is said to have cohomology type $(a, 0)$. Here we construct examples of free Z_p actions (p an odd prime) on certain of these spaces and give a complete description of possible fixed point sets.

INTRODUCTION

Let X be a CW complex with cohomology groups satisfying:

$$H^j(X; Z) = \begin{cases} Z & j = 0, n, 2n, 3n, \\ 0 & \text{otherwise.} \end{cases}$$

Let u_i generate $H^{in}(X; Z)$, $i = 0, 1, 2, 3$. X is said to be of type (a, b) if:

$$u_1^2 = au_2 \quad \text{and} \quad u_1u_2 = bu_3.$$

These spaces were first investigated by James [J] and Toda [T] in the late fifties. Let p be an odd prime. Supposing $b \neq 0 \pmod p$ then $X \approx_p S^n \times S^{2n}$ or $X \approx_p P^3(n)$ when $a = 0 \pmod p$ or $a \neq 0 \pmod p$ respectively. Note that for $a \neq 0 \pmod p$ n must be even. The nature of the fixed set has been studied in some detail for the case when Z_p acts on X and $b \neq 0 \pmod p$ [B, Chapter 7]. We will consider the case $b = 0 \pmod p$.

In §1 we study the structure of the fixed point set and obtain the following:

Theorem 1. *Let X be of type $(a, 0) \pmod p$ and let Z_p ~~act on~~ X, where p is an odd prime. If X is totally nonhomologous to zero in X_{Z_p} then for the fixed point set F:*

(a) *F has at most four components.*
(b) *If F has four components, each is acyclic and n is even.*
(c) *If F has three components then n is even and F is of the form, $F \approx_p S^{2m}VS^0VS^0$ (a bouquet).*
(d) *If F has two components then either $F \approx_p$ wedge of spheres or $F \approx_p$ disjoint union of a projective space $P^2(2m)$, $2m \leq n$ and a point.*

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(e) *If F has one component then $F \approx_p$ wedge of three spheres or $F \approx_p$ wedge of a projective space $P^2(2m)$, $2m \leq n$ and a sphere.*

Moreover, all these possibilities occur and if n is even then X is always totally nonhomologous to zero in X_{Z_p}.

In §2 we show that for n odd X need not be totally nonhomologous to zero by constructing free actions of Z_p on spaces of type $(0, 0)$ for certain odd n and certain primes p. It must be pointed out that this corrects results announced by the second author in [S] where it was asserted that Z_p cannot act freely on such spaces (also see R. Stong, review in Zentralblatt (572), May 1986, rev. #57020, p. 252).

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Here we prove Theorem 1. The methods, results, and notation that we use are those of [B, Chapter 7]. Cohomology will be taken with Z_p coefficients throughout this section and for convenience, will be suppressed from notation. Recall that a space Y is a mod p cohomology projective space of height h denoted $Y \approx_p P^h(n)$, if the mod p cohomology of Y is $Z_p[a]/a^{h+1}$, where a has degree n. We will say that Y is a $P^h(n)$ in this situation.

Because we are assuming that X is totally nonhomologous to zero in X_{Z_p}, we must have, by [B, Chapter 7], that $\Sigma \operatorname{rk} H^i(X) = 4 = \Sigma \operatorname{rk} H^i(F)$. It follows directly that F can have at most four components and examples can readily be constructed with F having from one to four components. Clearly if F has four components then each component is a cohomology point and $\chi(F) = 4$. Since $\chi(X) = 0$, if n is odd and p is an odd prime, then n must be even.

Now suppose F has three components. It is easy to see that one of the components must be a cohomology sphere and the others, cohomology points. However, if n is odd then $\chi(X) = 0$ while $\chi(F) \neq 0 \pmod p$ because p is odd. Hence n must be even. But then $\chi(X) = 4$ and the cohomology sphere component must be an even sphere. This establishes (c).

If F has two components and neither is a cohomology sphere then one component has the cohomology of a point while the other, for example F_0, satisfies the condition that $\Sigma \operatorname{rk} H^*(F_0) = 3$. If n is odd then F_0 has the cohomology of a wedge of spheres (for there are no cup products) and $F \approx_p F_0 V S^0$. If n is even then F_0 may be either a wedge of spheres or a projective space, $P^2(2m)$, $m \leq n$; as the following constructions show. By simply taking an appropriate wedge of Z_p actions on suitable spheres it is easy to produce actions with F_0, a wedge of spheres. Now by [B, Chapter 7, §4], for n even there is a Z_p action on some $P^2(n)$ with fixed set $P^2(k)$ (k even). Because n is even, Z_p can act freely on S^{n-1}, so consider the join $J = S^{n-1} * P^2(n)$ with fixed set $P^2(k)$. J has cohomology in dimensions $0, 2n, 3n$. Letting Z_p act on S^n with two fixed points we can form $X = S^n V J$ which provides an example of the type $(0, 0)$ having F_0 a $P^2(k)$. Of course if Z_p acts on S^{3n} with two fixed points then $X = P^2(n) V S^{3n}$ would be an example of a space of type $(a, 0)$ $(a \neq 0 \pmod p)$ with fixed set the disjoint union of a point and a projective space.

Finally if F is connected, examples may be constructed as above with fixed set, a wedge of spheres or a sphere and a projective space. If $a \neq 0 \pmod p$ then the fixed set cannot be a wedge of spheres.

As to the final remark of Theorem 1, if $p > 3$ and n even then $\Sigma \operatorname{rk} H^*(F) = \Sigma \operatorname{rk} H^*(X) = 4$ so X is totally nonhomologous to zero by [B, Chapter 7, 1.6]. If $p = 3$ and X is not totally nonhomologous to zero then by [B, Chapter 7, 2.2] the only possibility for F would be a cohomology point. But using rational coefficients we see from [B, Chapter 3, 2.4] that $\chi(X) = \chi(X/G) = 4$ and so from [B, Chapter 3, 4.3] we have $\chi(X^G) = \chi(X)$ which is a contradiction.

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Now we construct free Z_p actions on type $(0, 0)$ spaces. We begin with the complex $Y = S^{n-2} \times S^{2n-2}$ where n is odd. Let Z_p act freely on S^{n-2} and trivially on S^{2n-2} so that Z_p acts freely on Y. Now let $A = S^1 * Y$, the unreduced join, where S^1 (and so A) has a free Z_p action. Note that the top $(3n-4)$-dimensional cell of Y is attached by a Whitehead product which becomes homotopically trivial in A because we have in effect suspended twice. Thus the top-dimensional homology of A, $H_{3n-2}(A)$, is generated by a spherical class. Hence we can attach a free orbit of cells to A, producing a complex X with free Z_p action and having integral homology only in dimensions 0, n, $2n$, and $3n-1$. Now consider the following homology–homotopy diagram:

$$\begin{array}{ccccccc} & & \Pi_{3n-1}(X) & \xrightarrow{i_\#} & \Pi_{3n-1}(X, A) & \xrightarrow{\partial} & \Pi_{3n-2}(A) & \longrightarrow \\ & & \downarrow h & & \cong \downarrow h & & \downarrow h & \\ 0 & \longrightarrow & H_{3n-1}(X) & \underset{i_*}{\longrightarrow} & H_{3n-1}(X, A) & \underset{\partial}{\twoheadrightarrow} & H_{3n-2}(A) & \longrightarrow \end{array}$$

The maps h are Hurewicz homomorphisms. It is clear that $H_{3n-1}(X, A) \cong Z[Z_p]$. Since $H_{3n-2}(A) \cong Z$, $\operatorname{Im}(i_*) \cong I_\varepsilon$, the augmentation ideal. We want to show that $h^{-1}(\operatorname{Im}(i_*))$ is contained in $\operatorname{Im}(i_\#) = \ker(\partial)$ at least for certain values of n. Now A has the homotopy type $S^n V S^{2n} V S^{3n-2}$ and contains a subcomplex B of homotopy type $S^n V S^{2n}$. Under the Z_p action on A $\Pi_{3n-2}(A)$ becomes a $Z[Z_p]$ module; $H_{3n-2}(A)$ is a trivial $Z[Z_p]$ module, and the Hurewicz map $h: \Pi_{3n-2}(A) \to H_{3n-2}(A)$ is a surjective Z_p-map. Consider the diagram:

$$\begin{array}{ccccc} \Pi_{3n-2}(B) & \xrightarrow{i_\#} & \Pi_{3n-2}(A) & \longrightarrow & \Pi_{3n-2}(A, B) \\ & & \downarrow h & & \cong \downarrow h \\ 0 & \longrightarrow & H_{3n-2}(A) & \longrightarrow & H_{3n-2}(A, B) \end{array}$$

Because the right-hand vertical and bottom maps are injective it follows that $\ker(h) = \operatorname{Im}(i_\#)$. Now for suitable n and p, $\operatorname{Im}(i_\#)$ will have trivial Z_p

action. For example if $n = 3$ then $\Pi_7(B) = Z_2 + Z_2$ and $\text{Aut}(\Pi_7(B)) \cong S_3$ (symmetric group on 3 letters). Consequently if $p > 3$ then $\Pi_7(B)$ and hence $\text{Im}(i_\#)$ must have trivial action. Hence in these cases $\Pi_{3n-2}(A)$ will be a trivial $Z[Z_p]$ module. So $h^{-1}(\text{Im}_*)$ is contained in $\ker(\partial) = \text{Im}(i_\#)$, at least for certain n and p. This implies that $h: \Pi_{3n-1}(X) \to H_{3n-1}(X)$ is onto. So we may attach a free orbit of $3n$ cells to X to produce the complex W such that $H_{in}(W) = Z$ for $i = 0, 1, 2, 3$ and zero otherwise. This completes the construction. □

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