A NOTE ON EIGENVALUES OF HECKE OPERATORS ON SIEGEL MODULAR FORMS OF DEGREE TWO

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Abstract. Let $F$ be a cuspidal Hecke eigenform of even weight $k$ on $\text{Sp}_4(\mathbb{Z})$ with associated eigenvalues $\lambda_m$ ($m \in \mathbb{N}$). Under the assumption that its first Fourier-Jacobi coefficient does not vanish it is proved that the abscissa of convergence of the Dirichlet series $\sum_{m \geq 1} |\lambda_m| m^{-s}$ is less than or equal to $k$.

Introduction

Let $F$ be a cuspidal Hecke eigenform of weight $k \in 2\mathbb{N}$ on $\text{Sp}_4(\mathbb{Z})$ with associated eigenvalues $\lambda_m$ ($m \in \mathbb{N}$) and assume that its first Fourier-Jacobi coefficient does not vanish. Then we shall show ($\S$2) that the abscissa of convergence $\sigma_0$ of the Dirichlet series $\sum_{m \geq 1} |\lambda_m| m^{-s}$ is less than or equal to $k$. In an equivalent form, this means that $\sum_{1 \leq m \leq N} |\lambda_m| = O(e^{N^{k+\epsilon}})$ for every $\epsilon > 0$. Note that the (straightforward) estimate $\lambda_m = O(m^k)$ only implies $\sigma_0 \leq k + 1$, while (for $F$ not in the Maass space) the Ramanujan-Petersson conjecture would predict $\sigma_0 \leq k - \frac{1}{2}$.

Our result follows almost immediately from properties of Fourier-Jacobi coefficients of Siegel modular forms proved in [8]; however, it does not seem to have been noticed before.

1. Estimates for eigenvalues

For $k$ even denote by $S_k(\Gamma_2)$ the space of Siegel cusp forms of weight $k$ on $\Gamma_2 := \text{Sp}_4(\mathbb{Z})$ and write $T_k(m)$ ($m \in \mathbb{N}$) for the $m$th Hecke operator on $S_k(\Gamma_2)$. By definition,

$$T_k(m)F = m^{2k-3} \sum_{M \in \Gamma_2 \setminus O_2(m)} F|kM \quad (F \in S_k(\Gamma_2)),$$

where $M$ runs through a set of representatives for the action of $\Gamma_2$ by left-multiplication on the set $O_2(m)$ of integral $(4, 4)$-matrices that are symplectic similitudes with scale $m$ and as usual we have put

$$(F|kM)(Z) = \det(CZ + D)^{-k} F((AZ + B)(CZ + D)^{-1})$$

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(\( M = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \), \( Z \in \mathcal{H}_2 \) = Siegel upper half-space of degree 2).

Let \( F \in S_k(\Gamma_2) \) be a nonzero eigenfunction of all Hecke operators, and denote by \( \lambda_m \) the eigenvalue of \( F \) with respect to \( T_k(m) \). We let \( \sigma_0 \) be the abscissa of convergence of the Dirichlet series \( \sum_{m \geq 1} |\lambda_m| m^{-s} \). Note that by classical results on Dirichlet series, one has \( \sigma_0 = \inf \{ \alpha \in \mathbb{R} | \sum_{1 \leq m \leq N} |\lambda_m| = O(N^\alpha) \} \) (supposing that \( \sum_{m \geq 1} |\lambda_m| \) diverges).

Counting the number of left cosets in \( O_2(m) \) (cf. [1, 4, 9]) one easily obtains the estimate \( \lambda_m = O(m^k) \) and hence \( \sigma_0 \leq k + 1 \).

This can be improved to
\[
\lambda_m = O_e(m^{k-1/2+\epsilon}) \quad (\epsilon > 0)
\]
(hence \( \sigma_0 \leq k + \frac{1}{2} \)) if one combines on the one-hand side Kitaoka's estimate
\[
a(T) = O_e((\det T)^{k/2-1/4+\epsilon}) \quad (\epsilon > 0),
\]
where \( a(T) \) is the \( T \)th Fourier coefficient of \( F \) and \( T \) is any positive definite half-integral \((2, 2)\)-matrix (cf. [6]), and on the other hand Andrianov's results relating the spinor zeta function \( Z_F(s) \) of \( F \) to partial zeta functions of the form \( \sum_{m \geq 1} a(mT)m^{-s} \) (cf. [1]); recall that if one puts \( Z_{F, p}(X) := 1 - \lambda_pX + (\lambda_p^2 - \lambda_p^2 - p^{2k-4})X^2 - \lambda_p p^{2k-3}X^3 + p^{4k-6}X^4 \) \( (p \) a prime), then \( Z_F(s) = \prod_p Z_{F, p}(p^{-s})^{-1} = \zeta(2s - 2k + 4)^{-1} \sum_{m \geq 1} \lambda_m m^{-s} \) [1]. Also by [1], \( \tilde{Z}_F(s) = (2\pi)^{-2s}(s-k+2) Z_F(s) \) has a meromorphic continuation to \( \mathbb{C} \) and is invariant under \( s \mapsto 2k - 2 - s \).

If \( F \) is in the Maass subspace \( S_k(\Gamma_2)^* \subset S_k(\Gamma_2) \), then \( Z_F(s) = \zeta(s-k+1) \times \zeta(s-k+2)L_f(s) \) where \( f \) is an elliptic cusp form of weight \( 2k - 2 \) [2]. From this one can easily conclude that \( \lambda_m > 0 \), and therefore \( \sigma_0 = k \) for such \( F \). Note that by [3, 10] one has \( F \in S_k(\Gamma_2)^* \) iff \( \tilde{Z}_F(s) \) has poles.

On the other hand, for \( F \) in the orthogonal complement of the Maass space, one expects that the generalized Ramanujan-Petersson conjecture holds which predicts that the roots of the polynomial \( Z_{F, p}(X) \) are of absolute value \( p^{-k+3/2} \) for all \( p \).

The Ramanujan-Petersson conjecture would imply the estimate
\[
\lambda_m = O_e(m^{k-3/2+\epsilon}) \quad (\epsilon > 0)
\]
and hence \( \sigma_0 \leq k - \frac{1}{2} \), but at least at present a proof of it seems to be out of range.

2. Statement of result and proof

Recall that the function \( F \) has a Fourier-Jacobi expansion of the form
\[
F(Z) = \sum_{m \geq 1} \varphi_m(\tau, z)e^{2\pi i m \tau} \quad \left( Z = \left( \begin{array}{c} \tau \\ z \end{array} \right) \right),
\]
where the \( \varphi_m \)'s are Jacobi cusp forms of weight \( k \) and index \( m \) [2]. We shall
Theorem. Let \( F \in S_k(\Gamma_2) \) be a nonzero Hecke eigenform with eigenvalues \( \lambda_m \) w.r.t. \( T_k(m) \) and suppose that its first Fourier-Jacobi coefficient \( \varphi_1 \) does not vanish. Denote by \( \sigma_0 \) the abscissa of convergence of the Dirichlet series \( \sum_{m \geq 1} |\lambda_m| m^{-s} \). Then \( \sigma_0 \leq k \).

Remark. The condition "\( \varphi_1 \neq 0 \)" holds for (at least) all nonzero Hecke eigenforms of weight \( k \) with \( k \) in the range \( 10 \leq k \leq 32 \) (cf. [12]). Being optimistic, one could hope that it is always satisfied (cf. [11]).

Proof of Theorem. The assertion is a more or less immediate consequence of the results proved in [8]. Let \( J_{k, m}^{\text{cusp}} \) be the space of Jacobi cusp forms of weight \( k \) and index \( m \) and write \( \langle , , \rangle = \langle , , \rangle_{m, k} \) for the usual Petersson scalar product on \( J_{k, m}^{\text{cusp}} \) [2]. Let \( G \) be any function in the Maass space \( S_k(\Gamma_2)^* \) and write \( \psi_m \) for its \( m \)th Fourier-Jacobi coefficient. Then according to [5, 8] we have
\[
\langle \varphi_m, \psi_m \rangle = \langle \varphi_1, \psi_1 \rangle \lambda_m.
\]
Since \( \varphi_1 \neq 0 \) by assumption and the map \( G \mapsto \psi_1 \) is an isomorphism of \( S_k(\Gamma_2)^* \) onto \( J_{k, 1}^{\text{cusp}} \) [2] we can find \( G \in S_k(\Gamma_2)^* \) with \( \langle \varphi_1, \psi_1 \rangle \neq 0 \), and with this choice of \( G \) we can write
\[
\lambda_m = \langle \varphi_m, \psi_m \rangle / \langle \varphi_1, \psi_1 \rangle.
\]
By the Cauchy-Schwarz inequality we have
\[
|\langle \varphi_m, \psi_m \rangle| \leq \|\varphi_m\| \cdot \|\psi_m\| \leq \frac{1}{2}(\|\varphi_m\|^2 + \|\psi_m\|^2).
\]
On the other hand, as proved in [8], the Dirichlet series \( \sum_{m \geq 1} \|\varphi_m\|^2 m^{-s} \) (resp. \( \sum_{m \geq 1} \|\psi_m\|^2 m^{-s} \))—originally only defined in the half-plane \( \text{Re}(s) > k + 1 \)—have meromorphic continuations to \( \text{Re}(s) \geq k \) with simple poles at \( s = k \) as its only singularities. Therefore, by a theorem of Landau the abscissa of convergence of these Dirichlet series is equal to \( k \). Hence we deduce \( \sigma_0 \leq k \).

Remarks. (i) An interesting question seems to be whether the estimate \( \|\varphi_m\|^2 = O_{\varepsilon}(m^{k-1+\varepsilon}) \) would hold for any Hecke eigenform \( F \in S_k(\Gamma_2) \) (from the above discussion one sees that it is certainly satisfied for \( F \in S_k(\Gamma_2)^* \)). Note that under the condition "\( \varphi_1 \neq 0 \)" this would imply that \( \lambda_m = O_{\varepsilon}(m^{k-1+\varepsilon}) \). To answer the above question one naturally seems to be led to a closer study of the Fourier-Jacobi coefficients of the Poincaré-type series introduced by Klingen in [7].

(ii) S. Böcherer informs the author that very recently J. S. Li (using results of R. Howe) proved the estimate \( |\lambda_p| \leq 4p^{k-1} \) (\( p \) a prime).

References


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