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THERE ARE 2^c SYMMETRICALLY CONTINUOUS FUNCTIONS

MIROSLAV CHLEBÍK

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ABSTRACT. The purpose of this paper is to prove that the power of the set of symmetrically continuous real functions is 2^c (c is the power of the continuum). This surprisingly contrasts with the set of continuous (or Borel) real functions, the power of which is c.

1. INTRODUCTION

A function $f: \mathbb{R} \to \mathbb{R}$ is said to be symmetrically continuous at the point $x \in \mathbb{R}$ if

$$\lim_{h \to 0}[f(x+h) - f(x-h)] = 0.$$

This notion originates in the study of the pointwise convergence of trigonometric series. For more information and a rich bibliography, see the survey by L. Larson [5].

In the study of symmetrically continuous functions by itself the results of particular importance were proved by H. Fried [3] and D. Preiss [7]. They answered the questions stated by F. Hausdorff [4] about the set $\mathbb{R}\backslash C(f)$ of points where a symmetrically continuous (on $\mathbb{R}$) function f can be discontinuous. In fact, the set $\mathbb{R}\backslash C(f)$ is a Lebesgue null set of type F_σ. On the other hand, it can be uncountable.

The results mentioned above imply, in particular, that any symmetrically continuous function is a Lebesgue measurable function with the Baire property. The problem of whether it is Borel measurable or even of Baire class one has been stated several times [5, 6, 2]. The purpose of this paper is to give a negative answer to these questions. In fact, our theorem states that the power of the set of symmetrically continuous functions is 2^c, where c is the power of the continuum.

2. PRELIMINARIES AND RESULTS

Notation. Let A, B be subsets of $\mathbb{R}$. The symbols $2A$, $A+B$, and $A-B$ stand for the sets $\{2x: x \in A\}$, $\{(x+y): x \in A \text{ and } y \in B\}$, and $\{(x-y): x \in A \text{ and}$

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$y \in B\}$, respectively. The notation $A \backslash B$ stands for the set-theoretic difference of A and B. The symbol χ_A means the characteristic function of A and 2^A stands for the power set of A.

The set of centers of symmetry (of local symmetry) is denoted by A^* (A^*_{loc}, respectively), i.e.

$$A^* = \{x \in \mathbb{R}\colon \text{ whenever } h \in \mathbb{R} \text{ then } (x+h) \in A \text{ iff } (x-h) \in A\},$$
$$A^*_{\mathrm{loc}} = \{x \in \mathbb{R}\colon \text{ there exists } \delta = \delta(x) > 0 \text{ such that whenever } h \in (-\delta, \delta) \text{ then } (x+h) \in A \text{ iff } (x-h) \in A\}.$$

For any function $f\colon \mathbb{R} \to \mathbb{R}$ the symbol $C(f)$ stands for the set of continuity points of A.

Lemma 1. *Let $\sum_{n=1}^{\infty} \rho_n \cos(nx - \alpha_n)$ be a real trigonometric series with $\sum_{n=1}^{\infty} |\rho_n| = \infty$. Then the set*

$$\left\{x \in \mathbb{R}\colon \sum_{n=1}^{\infty} |\rho_n \cos(nx - \alpha_n)| < \infty\right\}$$

of points of its absolute convergence is a (Lebesgue) null set of type $\mathscr{F}_\sigma$.

The function $f\colon \mathbb{R} \to [0, 1]$ defined for every $x \in \mathbb{R}$ by the formula

$$f(x) = \lim_{m\to\infty} \left(1 + \sum_{n=1}^{m} |\rho_n \cos(nx - \alpha_n)|\right)^{-1}$$

is symmetrically continuous and upper semicontinuous.

Proof. The first part is the classical Denjoy-Luzin theorem, see e.g. [1, p. 173], the second one is proved in [7].

Lemma 2. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a symmetrically continuous function and let $A \subseteq \mathbb{R}$ be a set such that whenever $x \in \mathbb{R}$ either $\lim_{y\to x} f(y) = 0$ or $x \in A^*_{\mathrm{loc}}$. Then the function $f \cdot \chi_A$ is symmetrically continuous.*

Proof. Let $x \in \mathbb{R}$ be arbitrarily chosen. If $\lim_{y\to x} f(y) = 0$ we see that $\lim_{y\to x} f \cdot \chi_A(y) = 0$ and hence $f \cdot \chi_A$ is symmetrically continuous at x. If $x \in A^*_{\mathrm{loc}}$ we can choose a positive number $\delta = \delta(x)$ such that

$$\chi_A(x+h) = \chi_A(x-h) \quad \text{whenever } h \in (-\delta, \delta).$$

For such an h we have

$$f \cdot \chi_A(x+h) - f \cdot \chi_A(x-h) = [f(x+h) - f(x-h)] \cdot \chi_A(x+h)$$

and obviously $f \cdot \chi_A$ is symmetrically continuous at x.

Lemma 3. *Let $E \subseteq \mathbb{R}$ be an additive subgroup of $\mathbb{R}$ and let $G \subseteq E$ be a nonempty set such that the following implication holds:*

$$x \in G \quad \text{and} \quad y \in G \quad \text{and} \quad x \neq y \quad \text{imply} \quad \tfrac{1}{2}(x+y) \notin E.$$

Then the following assertions are true:

(i) $2E + H_1 \neq 2E + H_2$ *whenever* $\varnothing \neq H_i \subseteq G (i = 1, 2)$ *with* $H_1 \neq H_2$ *and*

(ii) $2E + H \subseteq E \subseteq [2E + H]^*$ *whenever* $\varnothing \neq H \subseteq G$.

Proof. (i) We assume to the contrary that we can find two nonempty subsets H_1, H_2 of G with $H_1 \backslash H_2 \neq \varnothing$ such that $2E + H_1 = 2E + H_2$. Then for a fixed $x \in H_1 \backslash H_2$ we have

$$x \in 2E + \{x\} \subseteq 2E + H_1 = 2E + H_2$$

and hence there exists $y \in H_2$ such that $x \in 2E + \{y\}$. Obviously $y \neq x$ and $\frac{1}{2}(x + y) \in E + \{y\} = E$ which contradicts the assumption made about E and G.

(ii) Let H be a nonempty subset of G. The first inclusion is trivial. To prove the second one we fix $x \in E$ and $h \in \mathbb{R}$ such that

$$(x + h) \in 2E + H.$$

Then we have

$$(x - h) = [2x - (x + h)] \in [2E - (2E + H)] = 2E + H,$$

hence $x \in [2E + H]^*$.

Theorem 1. *The power of the set of symmetrically continuous functions* $f\colon \mathbb{R} \to \mathbb{R}$ *is* 2^c.

Proof. Let $f\colon \mathbb{R} \to [0, 1]$ be defined by formula

$$f(x) = \lim_{m\to\infty} \left(1 + \sum_{n=1}^{m} \left|\frac{1}{n} \sin 3^n x\right|\right)^{-1}$$

for every $x \in \mathbb{R}$. We put

$$E = \left\{x \in \mathbb{R}\colon \sum_{n=1}^{\infty} \left|\frac{1}{n} \sin 3^n x\right| < \infty\right\} = \{x \in \mathbb{R}\colon f(x) \neq 0\}.$$

According to Lemma 1, f is symmetrically continuous, upper semicontinuous and E is a (Lebesgue) null set. Obviously, f is continuous at every point of $\mathbb{R} \backslash E$. Moreover, E is an additive subgroup of $\mathbb{R}$ since $0 \in E$ and

$$|\sin k(x \pm y)| \leq |\sin kx| + |\sin ky| \quad \text{whenever } x, y, k \in \mathbb{R}.$$

Let $\mathscr{H}$ be a basis of the linear space $\mathbb{R}$ over the field of rationals $\mathbb{Q}$ satisfying $1 \in \mathscr{H} \subseteq (0, 1]$.

We put $\Lambda = \mathscr{H} \backslash \{1\}$. Obviously the power of Λ is c. For any $\alpha \in \Lambda$ we define the infinite zero-one sequence $\{\mu_{\alpha,k}\}_{k=1}^{\infty}$ by the (unique) expansion

$$\alpha = \sum_{k=1}^{\infty} \mu_{\alpha,k} 2^{-k}.$$

For any pair $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$ we see that $(\alpha - \beta) \notin \mathbb{Q}$ and therefore, we can define the infinite strictly increasing sequence $\{t_{\alpha,\beta,r}\}_{r=1}^{\infty}$ of all natural numbers t with the property $\mu_{\alpha,t} \neq \mu_{\beta,t}$. We put $s_{\alpha,\beta,r} = t_{\alpha,\beta,2r}$ and then we see that

$$\sum_{k=1}^{s_{\alpha,\beta,r}-1} (\mu_{\alpha,k} + \mu_{\beta,k}) \equiv 1 \pmod 2 \tag{1}$$

and

$$(\mu_{\alpha, s_{\alpha,\beta,r}} + \mu_{\beta, s_{\alpha,\beta,r}}) = 1 \tag{2}$$

for every $r = 1, 2, 3, \ldots$.

For any $\alpha \in \Lambda$ we define the real number a_α by the formula

$$a_\alpha = \pi \cdot \left[\sum_{k=1}^{\infty} \mu_{\alpha,k} \cdot 3^{-(2^k+1)} \right]$$

and we put

$$G = \{a_\alpha : \alpha \in \Lambda\}.$$

It is easy to see that $a_\alpha \neq a_\beta$ whenever $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$. In particular, the power of G is c.

We shall prove that

$$G \subseteq E \tag{3}$$

and

$$\tfrac{1}{2}(x+y) \notin E \quad \text{whenever } x, y \in G \text{ with } x \neq y. \tag{4}$$

We first assume that (3) and (4) are proved. For every nonempty set $H \subseteq G$ we define the function $f_H : \mathbb{R} \to [0, 1]$ by the formula

$$f_H = f \cdot \chi_{(2E+H)}.$$

According to Lemma 2 and Lemma 3(ii) the function f_H is symmetrically continuous. Moreover

$$\{x \in \mathbb{R} : f_H(x) \neq 0\} = 2E + H$$

and therefore, by Lemma 3(i) $f_{H_1} \neq f_{H_2}$ holds whenever $\emptyset \neq H_i \subseteq G$ $(i = 1, 2)$ with $H_1 \neq H_2$. Hence the set

$$\{f_H : \emptyset \neq H \subseteq G\}$$

is a set of power 2^c and any of its elements is a symmetrically continuous function. This easily implies our theorem.

Now it is sufficient to prove (3) and (4). To prove (3) we fix an arbitrary $a_\alpha \in G$. Let s, n be positive integers such that $2^{s-1} < n \leq 2^s$. Then

$$3^n a_\alpha / \pi = 3^n \sum_{k=1}^{\infty} \mu_{\alpha,k} 3^{-(2^k+1)} \equiv c(s, n) \pmod 1,$$

where

$$c(s, n) = 3^n \sum_{k=s}^{\infty} \mu_{\alpha,k} \cdot 3^{-(2^k+1)} = 3^{n-2^s} \sum_{k=s}^{\infty} \mu_{\alpha,k} 3^{-(2^k-2^s+1)}.$$

The following inequalities are obvious

$$0 \leq c(s, n) < 3^{n-2^s} \sum_{l=1}^{\infty} 3^{-l} = \frac{1}{2} 3^{n-2^s} \qquad \left(\leq \frac{1}{2}\right)$$

and therefore,

$$|\sin 3^n a_\alpha| = |\sin \pi c(n, s)| \le \pi c(n, s) < \tfrac{\pi}{2} 3^{n-2^s}.$$

It easily implies that the inequalities

$$\sum_{n=2^{s-1}+1}^{2^s} \frac{1}{n}|\sin 3^n a_\alpha| \le 2^{1-s} \sum_{n=2^{s-1}+1}^{2^s} |\sin 3^n a_\alpha| < 3\pi 2^{-s-1}$$

hold for every positive integer s. Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n}|\sin 3^n a_\alpha| < 1 + 3\pi \sum_{s=1}^{\infty} 2^{-s-1} = 1 + \frac{3}{2}\pi < \infty$$

and hence $a_\alpha \in E$.

To prove (4) we fix $\alpha, \beta \in \Lambda$ with $\alpha \ne \beta$ and we put

$$b = \tfrac{1}{2}(a_\alpha + a_\beta).$$

We shall prove that $b \notin E$. Let s, n be positive integers such that

$$2^{s-1} < n \le 2^s.$$

Then

$$3^n \frac{2b}{\pi} = 3^n \sum_{k=1}^{\infty} (\mu_{\alpha,k} + \mu_{\beta,k}) \cdot 3^{-(2^k+1)} \equiv P(s) + Q(s, n) \pmod 2, \tag{5}$$

where

$$P(s) = \sum_{k=1}^{s-1} (\mu_{\alpha,k} + \mu_{\beta,k}),$$

$$Q(s, n) = 3^n \sum_{k=s}^{\infty} (\mu_{\alpha,k} + \mu_{\beta,k}) \cdot 3^{-(2^k+1)}.$$

We fix any pair of positive integers r, n such that

$$2^{s_{\alpha,\beta,r}-1} < n \le 2^{s_{\alpha,\beta,r}}.$$

Recalling (2) we see that the following relations hold:

$$\begin{aligned} 0 &< Q(s_{\alpha,\beta,r}, n) \\ &\le 3^{n-2^{s_{\alpha,\beta,r}}} \left[3^{-1} + \sum_{k=s_{\alpha,\beta,r}+1}^{\infty} 2 \cdot 3^{2^{s_{\alpha,\beta,r}} - (2^k+1)} \right] \\ &< 3^{n-2^{s_{\alpha,\beta,r}}} \left[3^{-1} + \sum_{l=2}^{\infty} 2 \cdot 3^{-l} \right] = \frac{2}{3} 3^{n-2^{s_{\alpha,\beta,r}}} \le \frac{2}{3}. \end{aligned} \tag{6}$$

From (1) we see that $P(s_{\alpha,\beta,r}) \equiv 1 \pmod 2$ and hence by (5)

$$3^n b = \frac{\pi}{2}\left(3^n \frac{2b}{\pi}\right) \equiv \frac{\pi}{2}[1 + Q(s_{\alpha,\beta,r}, n)] \pmod \pi. \tag{7}$$

By (6) we see that

$$\frac{\pi}{2}[1 + Q(s_{\alpha,\beta,r}, n)] \in \left(\frac{\pi}{2}, \frac{5\pi}{6}\right) \tag{8}$$

and from (7) and (8) we easily get that $|\sin 3^n b| > \frac{1}{2}$. Therefore

$$\sum_{n=2^{s_{\alpha,\beta,r}-1}+1}^{2^{s_{\alpha,\beta,r}}} \frac{1}{n}|\sin 3^n b| > 2^{-s_{\alpha,\beta,r}} \sum_{n=2^{s_{\alpha,\beta,r}-1}+1}^{2^{s_{\alpha,\beta,r}}} \frac{1}{2} = \frac{1}{4}$$

holds for any positive integer r and

$$\sum_{n=1}^{\infty} \frac{1}{n}|\sin 3^n b| = \infty .$$

Hence $b \notin E$ and (4) holds. This completes the proof.

Corollary 1. *There exists a symmetrically continuous function* $f\colon \mathbb{R} \to \mathbb{R}$ *that is not Borel measurable.*

Proof. The power of the set of all Borel measurable functions $f\colon \mathbb{R} \to \mathbb{R}$ is c.

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Institute of Applied Mathematics, Comenius University, Bratislava, Czechoslovakia

Current address: Department of Mathematics, Comenius University, 84215 Bratislava, Czechoslovakia