THERE ARE $2^c$ SYMMETRICALLY CONTINUOUS FUNCTIONS

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Abstract. The purpose of this paper is to prove that the power of the set of symmetrically continuous real functions is $2^c$ (c is the power of the continuum). This surprisingly contrasts with the set of continuous (or Borel) real functions, the power of which is c.

1. Introduction

A function $f: \mathbb{R} \to \mathbb{R}$ is said to be symmetrically continuous at the point $x \in \mathbb{R}$ if

$$\lim_{h \to 0} [f(x+h) - f(x-h)] = 0.$$  

This notion originates in the study of the pointwise convergence of trigonometric series. For more information and a rich bibliography, see the survey by L. Larson [5].

In the study of symmetrically continuous functions by itself the results of particular importance were proved by H. Fried [3] and D. Preiss [7]. They answered the questions stated by F. Hausdorff [4] about the set $\mathbb{R} \setminus C(f)$ of points where a symmetrically continuous (on $\mathbb{R}$) function $f$ can be discontinuous. In fact, the set $\mathbb{R} \setminus C(f)$ is a Lebesgue null set of type $F_\sigma$. On the other hand, it can be uncountable.

The results mentioned above imply, in particular, that any symmetrically continuous function is a Lebesgue measurable function with the Baire property. The problem of whether it is Borel measurable or even of Baire class one has been stated several times [5, 6, 2]. The purpose of this paper is to give a negative answer to these questions. In fact, our theorem states that the power of the set of symmetrically continuous functions is $2^c$, where c is the power of the continuum.

2. Preliminaries and results

Notation. Let $A, B$ be subsets of $\mathbb{R}$. The symbols $2A$, $A+B$, and $A-B$ stand for the sets $\{2x: x \in A\}$, $\{(x+y): x \in A \text{ and } y \in B\}$, and $\{(x-y): x \in A \text{ and } y \in B\}$, respectively.

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The notation $A \setminus B$ stands for the set-theoretic difference of $A$ and $B$. The symbol $\chi_A$ means the characteristic function of $A$ and $2^A$ stands for the power set of $A$.

The set of centers of symmetry (of local symmetry) is denoted by $A^*$ ($A^*_{\text{loc}}$, respectively), i.e.

$$A^* = \{ x \in \mathbb{R} : \text{whenever } h \in \mathbb{R} \text{ then } (x + h) \in A \iff (x - h) \in A \},$$

$$A^*_{\text{loc}} = \{ x \in \mathbb{R} : \text{there exists } \delta = \delta(x) > 0 \text{ such that whenever } h \in (-\delta, \delta) \text{ then } (x + h) \in A \iff (x - h) \in A \}.$$

For any function $f : \mathbb{R} \to \mathbb{R}$ the symbol $C(f)$ stands for the set of continuity points of $A$.

**Lemma 1.** Let $\sum_{n=1}^{\infty} |\rho_n| = \infty$. Then the set

$$\left\{ x \in \mathbb{R} : \sum_{n=1}^{\infty} |\rho_n \cos(nx - \alpha_n)| < \infty \right\}$$

is a (Lebesgue) null set of type $\mathcal{F}_\sigma$.

**The function**

$$f(x) = \lim_{m \to \infty} \left( 1 + \sum_{n=1}^{m} |\rho_n \cos(nx - \alpha_n)| \right)^{-1}$$

**is symmetrically continuous and upper semicontinuous.**

**Proof.** The first part is the classical Denjoy-Luzin theorem, see e.g. [1, p. 173], the second one is proved in [7].

**Lemma 2.** Let $f : \mathbb{R} \to \mathbb{R}$ be a symmetrically continuous function and let $A \subseteq \mathbb{R}$ be a set such that whenever $x \in \mathbb{R}$ either $\lim_{y \to x} f(y) = 0$ or $x \in A^{*}_{\text{loc}}$. Then the function $f \cdot \chi_A$ is symmetrically continuous.

**Proof.** Let $x \in \mathbb{R}$ be arbitrarily chosen. If $\lim_{y \to x} f(y) = 0$ we see that $\lim_{y \to x} f \cdot \chi_A(y) = 0$ and hence $f \cdot \chi_A$ is symmetrically continuous at $x$. If $x \in A^{*}_{\text{loc}}$ we can choose a positive number $\delta = \delta(x)$ such that

$$\chi_A(x + h) = \chi_A(x - h) \text{ whenever } h \in (-\delta, \delta).$$

For such an $h$ we have

$$f \cdot \chi_A(x + h) - f \cdot \chi_A(x - h) = [f(x + h) - f(x - h)] \cdot \chi_A(x + h)$$

and obviously $f \cdot \chi_A$ is symmetrically continuous at $x$.

**Lemma 3.** Let $E \subseteq \mathbb{R}$ be an additive subgroup of $\mathbb{R}$ and let $G \subseteq E$ be a nonempty set such that the following implication holds:

$$x \in G \text{ and } y \in G \text{ and } x \neq y \text{ imply } \frac{1}{2}(x + y) \not\in E.$$  

**Then the following assertions are true:**

(i) $2E + H_1 \neq 2E + H_2$ whenever $\emptyset \neq H_i \subseteq G(i = 1, 2)$ with $H_1 \neq H_2$ and

(ii) $2E + H \subseteq E \subseteq [2E + H]^*$ whenever $\emptyset \neq H \subseteq G$. 
Proof. (i) We assume to the contrary that we can find two nonempty subsets \( H_1, H_2 \) of \( G \) such that \( 2E + H_1 = 2E + H_2 \). Then for a fixed \( x \in H_1 \setminus H_2 \) we have
\[
x \in 2E + \{x\} \subseteq 2E + H_1 = 2E + H_2
\]
and hence there exists \( y \in H_2 \) such that \( x \in 2E + \{y\} \). Obviously \( y \neq x \) and \( \frac{1}{2}(x + y) \in E + \{y\} = E \) which contradicts the assumption made about \( E \) and \( G \).

(ii) Let \( H \) be a nonempty subset of \( G \). The first inclusion is trivial. To prove the second one we fix \( x \in E \) and \( h \in \mathbb{R} \) such that
\[
(x + h) \in 2E + H.
\]
Then we have
\[
(x - h) = [2x - (x + h)] \in [2E - (2E + H)] = 2E + H^*.
\]

Theorem 1. The power of the set of symmetrically continuous functions \( f: \mathbb{R} \to \mathbb{R} \) is \( 2^c \).

Proof. Let \( f: \mathbb{R} \to [0, 1] \) be defined by formula
\[
f(x) = \lim_{m \to \infty} \left( 1 + \sum_{n=1}^{m} \frac{1}{n} \sin 3^n x \right)^{-1}
\]
for every \( x \in \mathbb{R} \). We put
\[
E = \left\{ x \in \mathbb{R}: \sum_{n=1}^{\infty} \frac{1}{n} \sin 3^n x < \infty \right\} = \{ x \in \mathbb{R}: f(x) \neq 0 \}.
\]
According to Lemma 1, \( f \) is symmetrically continuous, upper semicontinuous and \( E \) is a (Lebesgue) null set. Obviously, \( f \) is continuous at every point of \( \mathbb{R} \setminus E \). Moreover, \( E \) is an additive subgroup of \( \mathbb{R} \) since \( 0 \in E \) and
\[
|\sin k(x \pm y)| \leq |\sin kx| + |\sin ky| \quad \text{whenever} \ x, y, k \in \mathbb{R}.
\]
Let \( \mathcal{H} \) be a basis of the linear space \( \mathbb{R} \) over the field of rationals \( \mathbb{Q} \) satisfying \( 1 \in \mathcal{H} \subseteq (0, 1] \).

We put \( \Lambda = \mathcal{H} \setminus \{1\} \). Obviously the power of \( \Lambda \) is \( c \). For any \( \alpha \in \Lambda \) we define the infinite zero-one sequence \( \{\mu_{\alpha, k}\}_{k=1}^{\infty} \) by the (unique) expansion
\[
\alpha = \sum_{k=1}^{\infty} \mu_{\alpha, k} 2^{-k}.
\]
For any pair \( \alpha, \beta \in \Lambda \) with \( \alpha \neq \beta \) we see that \( (\alpha - \beta) \notin \mathbb{Q} \) and therefore, we can define the infinite strictly increasing sequence \( \{t_{\alpha, \beta, r}\}_{r=1}^{\infty} \) of all natural numbers \( t \) with the property \( \mu_{\alpha, t} \neq \mu_{\beta, t} \). We put \( s_{\alpha, \beta, r} = t_{\alpha, \beta, 2r} \) and then we see that
\[
\sum_{k=1}^{s_{\alpha, \beta, r}-1} (\mu_{\alpha, k} + \mu_{\beta, k}) \equiv 1 \pmod{2} \quad (1)
\]
and

\[(2) \quad (\mu_{\alpha,s_{\beta,r}} + \mu_{\beta,s_{\alpha,r}}) = 1\]

for every \( r = 1, 2, 3, \ldots \).

For any \( \alpha \in \Lambda \) we define the real number \( a_\alpha \) by the formula

\[a_\alpha = \pi \cdot \left[ \sum_{k=1}^{\infty} \mu_{\alpha,k} \cdot 3^{-(2^k+1)} \right]\]

and we put

\[G = \{a_\alpha : \alpha \in \Lambda\}.

It is easy to see that \( a_\alpha \neq a_\beta \) whenever \( \alpha, \beta \in \Lambda \) with \( \alpha \neq \beta \). In particular, the power of \( G \) is \( \mathcal{c} \).

We shall prove that

\[(3) \quad G \subseteq E\]

and

\[(4) \quad \frac{1}{2}(x+y) \notin E \quad \text{whenever} \quad x, y \in G \text{ with } x \neq y.\]

We first assume that (3) and (4) are proved. For every nonempty set \( H \subseteq G \) we define the function \( f_H : \mathbb{R} \rightarrow [0, 1] \) by the formula

\[f_H = f \cdot \chi_{(2E+H)}.\]

According to Lemma 2 and Lemma 3(ii) the function \( f_H \) is symmetrically continuous. Moreover

\[{\{x \in \mathbb{R} : f_H(x) \neq 0\} = 2E + H}\]

and therefore, by Lemma 3(i) \( f_{H_1} \neq f_{H_2} \) holds whenever \( \emptyset \neq H_i \subseteq G \) \( (i = 1, 2) \) with \( H_1 \neq H_2 \). Hence the set

\[{\{f_H : \emptyset \neq H \subseteq G\}\}

is a set of power \( 2^c \) and any of its elements is a symmetrically continuous function. This easily implies our theorem.

Now it is sufficient to prove (3) and (4). To prove (3) we fix an arbitrary \( a_\alpha \in G \). Let \( s, n \) be positive integers such that \( 2^{s-1} < n \leq 2^s \). Then

\[3^n a_\alpha / \pi = 3^n \sum_{k=1}^{\infty} \mu_{\alpha,k} \cdot 3^{-(2^k+1)} \equiv c(s, n) \quad (\text{mod } 1),\]

where

\[c(s, n) = 3^n \sum_{k=s}^{\infty} \mu_{\alpha,k} \cdot 3^{-(2^k+1)} = 3^{n-2^s} \sum_{k=s}^{\infty} \mu_{\alpha,k} \cdot 3^{-(2^k-2^s+1)}.

The following inequalities are obvious

\[0 \leq c(s, n) < 3^{n-2^s} \sum_{l=1}^{\infty} 3^{-l} = \frac{1}{2} 3^{n-2^s} \quad \left( \leq \frac{1}{2} \right)\]
and therefore,
\[ |\sin 3^n a| = |\sin \pi c(n, s)| \leq \pi c(n, s) < \frac{\pi}{3} 3^{n-2}. \]

It easily implies that the inequalities
\[ \sum_{n=2s+1}^{2s+2} \frac{1}{n} |\sin 3^n a| \leq 2^{1-s} \sum_{n=2s+1}^{2s+2} |\sin 3^n a| < 3 \pi 2^{-s-1} \]
hold for every positive integer \( s \). Therefore,
\[ \sum_{n=1}^{\infty} \frac{1}{n} |\sin 3^n a| < 1 + 3 \pi \sum_{s=1}^{\infty} 2^{-s-1} = 1 + \frac{3}{2} \pi < \infty \]
and hence \( a \in E \).

To prove (4) we fix \( \alpha, \beta \in \Lambda \) with \( \alpha \neq \beta \) and we put
\[ b = \frac{1}{2} (a_\alpha + a_\beta). \]

We shall prove that \( b \not\in E \). Let \( s, n \) be positive integers such that
\[ 2^{s-1} < n \leq 2^s. \]

Then
\[ 3^n \frac{2b}{\pi} = 3^n \sum_{k=1}^{\infty} (\mu_\alpha,k + \mu_\beta,k) \cdot 3^{-(2^k+1)} \equiv P(s) + Q(s, n) \quad (\text{mod } 2), \]
where
\[ P(s) = \sum_{k=1}^{s-1} (\mu_\alpha,k + \mu_\beta,k), \]
\[ Q(s, n) = 3^n \sum_{k=s}^{\infty} (\mu_\alpha,k + \mu_\beta,k) \cdot 3^{-(2^k+1)}. \]

We fix any pair of positive integers \( r, n \) such that
\[ 2^{s-r-1} < n \leq 2^{s-r}. \]

Recalling (2) we see that the following relations hold:
\[ 0 < Q(s, \beta, r, n) \]
\[ \leq 3^{n-2^{s-r}} \left[ \frac{1}{3} + \sum_{k=s-r+1}^{\infty} 2 \cdot 3^{2^{s-r}-(2^k+1)} \right] \]
\[ < 3^{n-2^{s-r}} \left[ \frac{1}{3} + \sum_{l=2}^{\infty} 2 \cdot 3^{-l} \right] = \frac{2}{3} 3^{n-2^{s-r}} \leq \frac{2}{3}. \]

From (1) we see that \( P(s, \beta, r) \equiv 1 \) (mod 2) and hence by (5)
\[ 3^n b = \frac{\pi}{2} \left( 3^n \frac{2b}{\pi} \right) \equiv \frac{\pi}{2} \left[ 1 + Q(s, \beta, r, n) \right] \quad (\text{mod } \pi). \]
By (6) we see that
\[ \frac{\pi}{2} [1 + Q(s, \beta, r, n)] \in \left( \frac{\pi}{2}, \frac{5\pi}{6} \right) \]
and from (7) and (8) we easily get that \(|\sin 3^n b| > \frac{1}{2}|. Therefore

\[ \sum_{n=2^s, \beta, r^{-1}+1}^{2^s, \beta, 2^n} \frac{1}{n} |\sin 3^n b| > 2^{-s} \sin b \sum_{n=2^s, \beta, r^{-1}+1}^{2^s, \beta, 2^n} \frac{1}{2} = \frac{1}{4} \]
holds for any positive integer \( r \) and

\[ \sum_{n=1}^{\infty} \frac{1}{n} |\sin 3^n b| = \infty. \]

Hence \( b \notin E \) and (4) holds. This completes the proof.

**Corollary 1.** There exists a symmetrically continuous function \( f: \mathbb{R} \to \mathbb{R} \) that is not Borel measurable.

**Proof.** The power of the set of all Borel measurable functions \( f: \mathbb{R} \to \mathbb{R} \) is \( c \).

**References**


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