PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 113, Number 3, November 1991

NONREALIZABILITY OF SUBALGEBRAS OF $\mathfrak{A}^*$

STANLEY O. KOCHMAN

(Communicated by Frederick R. Cohen)

ABSTRACT. At the prime two, the dual of the Steenrod algebra is a polynomial algebra in generators ξ_n, $n \geq 1$. The Eilenberg-Mac Lane spectrum $K(Z_2)$ has homology $Z_2[\xi_n | n \geq 1]$, the Brown-Peterson spectrum BP has homology $Z_2[\xi_n^2 | n \geq 1]$, and the symplectic Thom spectrum MSp has homology $Z_2[\xi_n^4 | n \geq 1] \otimes \mathfrak{S}$. In this paper, we show that there is no spectrum B_k with $H_* B_k = Z_2[\xi_n^{2^k} | n \geq 1]$ for $k \geq 2$.

In this paper, all spectra are localized at the prime two, and all coefficients are Z_2. All our spectra E have units $\mu: S \to E$, and all our ring spectra have a homotopy unit, are homotopy associative, and are homotopy commutative. Let $\mathfrak{A}^* = Z_2[\xi_n | n \geq 1]$ denote the dual of the Steenrod algebra. Recall [2, 7] that as algebras and $\mathfrak{A}^*$-comodules, $H_* KZ_2 = Z_2[\xi_n | n \geq 1]$, $H_* \mathrm{BP} = Z_2[\xi_n^2 | n \geq 1]$, and $H_* \mathrm{MSp} = Z_2[\xi_n^4 | n \geq 1] \otimes \mathfrak{S}$, where $\mathfrak{S} = Z_2[V_m | m \neq 2^q - 1]$. The V_m are $\mathfrak{A}^*$-primitive elements of degree $4m$. In this paper we show that for $k \geq 2$ there is no ring spectrum B_k such that $H_* B_k = Z_2[\xi_n^{2^k} | n \geq 1]$ as algebras and $\mathfrak{A}^*$-comodules. For $k \geq 4$, we prove the stronger result that there is no spectrum $\mathfrak{B}_k$ such that $H_* \mathfrak{B}_k = Z_2[\xi_n^{2^k} | n \geq 1] \otimes \mathfrak{S}_k$ as $\mathfrak{A}^*$-comodules, where $\mathfrak{S}_k$ is a set of $\mathfrak{A}^*$-primitive elements. Of course, MSp is an example of a $\mathfrak{B}_2$. We cannot determine whether any $\mathfrak{B}_3$ exist. If spectra of the type $\mathfrak{B}_k$, $k \geq 3$, had existed, they would have defined generalized Adams spectral sequences which would have been efficient methods for computing π_*^S. (For example, see [6] for a description of the MSp-Adams-Novikov spectral sequence for π_*^S.)

Assume that B_k exists with $k \geq 2$. Consider the Adams spectral sequence:

$$\text{(A)} \qquad E_2^{n,t} = \mathrm{Ext}_{\mathfrak{A}}^t (H^* B_k, Z_2)_n \Rightarrow \pi_* B_k .$$

Note that since B_k may not be a ring spectrum, the Adams spectral sequence (A) may not have a multiplicative structure. However, $H_* B_k = Z_2[\xi_n^{2^k} | n \geq 1]$ is

Received by the editors March 9, 1990.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 55P42; Secondary 55N22, 55S10, 55T15.

This research was partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

a sub-Hopf algebra of $\mathfrak{A}^*$. Let $T(k)$ denote the truncated polynomial algebra $Z_2[\xi_n|n \geq 1]/(\xi_n^{2^k}|n \geq 1)$. By the change of rings theorem of A. Liulevicius [7]

$$\begin{aligned}\mathrm{Ext}_{\mathfrak{A}}(H^*B_k, Z_2) &\cong \mathrm{Cotor}_{\mathfrak{A}^*}(H_*B_k, Z_2)\\ &= \mathrm{Cotor}_{\mathfrak{A}}(Z_2[\xi_n^{2^k}|n \geq 1], Z_2) \cong \mathrm{Cotor}_{T(k)}(Z_2, Z_2).\end{aligned}$$

We use the May spectral sequence [8] to study $\mathrm{Cotor}_{T(k)}(Z_2, Z_2)$:

$$\text{(B)} \qquad {}_ME_2^{n,t} = \mathrm{Cotor}^t_{E^0T(k)}(Z_2, Z_2)_n \Rightarrow \mathrm{Cotor}^t_{T(k)}(Z_2, Z_2)_n.$$

$E^0T(k)$ is the associated graded algebra of $T(k)$ induced by the coproduct filtration. Let $\mathfrak{C}^*T(k)$ denote the cobar construction of $T(k)$. The following lemma describes a DGA algebra ${}_ME_1$ whose homology is ${}_ME_2$.

Lemma 1. *Let* $h_{nj} = [\xi_n^{2^j}] \in \mathfrak{C}^{2^{n+j}-2^j-1,1}T(k)$. *Then*

$${}_ME_1 = Z_2[h_{nj}|n \geq 1 \text{ and } k > j \geq 0].$$

Moreover, $d_1(h_{nj}) = \sum_{t=1}^{k-j-1} h_{n-t,j+t}h_{tj}$ *for* $0 \leq j \leq k-2$, *and* $d_1(h_{n,k-1}) = 0$.

Proof. Let $E(S)$, $P(S)$, $\Gamma(S)$ denote the exterior algebra, polynomial algebra, and divided polynomial algebra, respectively, on the set S, and let $V(\mathfrak{L})$ denote the universal enveloping algebra of the restricted Lie algebra $\mathfrak{L}$ [10]. Let $\xi_{nj} = \{\xi_n^{2^j}\} \in E^0T(k)$. Then $E^0T(k) = E(\xi_{nj}|n \geq 1,\ k > j \geq 0)$ with $\widetilde{\Delta}(\xi_{nj}) = \sum_{t=1}^{k-j-1} \xi_{n-t,t+j} \otimes \xi_{tj}$. Thus, $[E^0T(k)]^* = V(\mathfrak{L})$, where $\mathfrak{L}$ is the restricted Lie algebra with Z_2-basis $\{\xi^*_{nj}|n \geq 1,\ k > j \geq 0\}$, zero restriction, and Lie bracket $[\xi^*_{mi}, \xi^*_{nj}]$ equal to $\xi^*_{m+n,i}$, $\xi^*_{m+n,j}$, 0 if $m+i=j$, $n+j=i$, $m+i \neq j$ and $n+j \neq i$, respectively. By [9, Remark 10], there is a differential on the Z_2-coalgebra $X = \Gamma(s\mathfrak{L}) \otimes V(\mathfrak{L})$ making X a free $[E^0T(k)]^*$-resolution of Z_2. Thus,

$$\begin{aligned}{}_ME_2 &= H_*[\mathrm{Hom}_{V(\mathfrak{L})}(X, Z_2)]\\ &\cong H_*[\mathrm{Hom}_{Z_2}(\Gamma(s\mathfrak{L}), Z_2)] \cong H_*P[(s\mathfrak{L})^*]\\ &\cong H_*(Z_2[h_{nj}|n \geq 1,\ k > j \geq 0]),\end{aligned}$$

where $h_{nj} = (s\xi^*_{nj})^*$ is represented by $[\xi_n^{2^j}]$ in the cobar construction. Clearly d_1, being induced by $\widetilde{\Delta}(\xi_{nj})$, is given by $d_1(h_{nj}) = \sum_{t=1}^{k-j-1} h_{n-t,j+t}h_{tj}$ for $0 \leq j \leq k-1$. The situation is analogous to that of [8, Chapter 2, §3] and [5, §1]. □

Lemma 2. *For* $k \geq 4$, *there is a nonzero element* $h_0h_{k-1}^2 \in E_2^{2^k-2,3}$.

Proof. Clearly $h_{10}h_{1,k-1}^2$ is a nonzero infinite cycle in ${}_ME_1^{2^k-2,3}$. So ${}_ME_1^{2^k-1,2} = Z_2\{h_{20}h_{k-1,1}, h_{11}h_{k0}\}$, and $d_1(h_{20}h_{k-1,1})$, $d_1(h_{11}h_{k0})$ contains $h_{20}h_{21}h_{k-3,3}$, $h_{11}h_{30}h_{k-3,3}$, respectively, as a nonzero summand because $k \geq 4$. Thus,

${}_M E_2^{2^k-1,2} = 0$, $h_{10}h_{1,k-1}^2$ is nonzero in ${}_M E_\infty^{2^k-2,3}$ and defines a nonzero element $h_0 h_{k-1}^2$ of $E_2^{2^k-2,3}$. □

Theorem 3. *Spectra $\mathfrak{B}_k$ and B_k do not exist for $k \geq 4$.*

Proof. Consider the unit map $\mu: S \to B_k$. Let the h_k be represented by the $h_{1,k}$ in the May spectral sequences. By Adams [1], $d_2(h_k) = h_0 h_{k-1}^2$ for $k \geq 4$ in the Adams spectral sequence for $\pi_* S$:

$$\text{(C)} \qquad E_2^{n,t} = \mathrm{Ext}_{\mathfrak{A}}^t(Z_2, Z_2)_n \Rightarrow \pi_* S.$$

μ induces a map of spectral sequences $\{\mu_r\}$ between the Adams spectral sequences (C) and (A). Observe that μ_2 is induced by the map of algebras $\mu_*: Z_2 \to H_* B_k = Z_2[\xi_m^{2^k} | m \geq 1]$, and thus μ_2 is an algebra homomorphism. Clearly $\mu_2(h_i) = h_i$ for $0 \leq i \leq k-1$, and $\mu_2(h_k) = 0$. Thus, $h_0 h_{k-1}^2$ must be zero in E_2 of the Adams spectral sequence (A), contradicting Lemma 2. Therefore, B_k cannot exist for $k \geq 4$. The E_2-term of the Adams spectral sequence for $\pi_* \mathfrak{B}_k$ equals $\mathrm{Cotor}_{T(k)}(Z_2, Z_2) \otimes \mathfrak{S}_k$. Thus, the above argument, with the Adams spectral sequence for $\pi_* \mathfrak{B}_k$ replacing the Adams spectral sequence (A), shows that $\mathfrak{B}_k$ cannot exist for $k \geq 4$. □

Theorem 4. *A ring spectrum B_2 does not exist.*

Proof. The E_2-term of the Adams spectral sequence (A) is $\mathrm{Cotor}_{T(2)}(Z_2, Z_2)$, which is computed in [5, §3], where $T(2)$ is called B. In the notation of [5, §3], $P(0, 1) \in E_2^{7,2}$ is a nonzero infinite cycle because $E_2^{8,0} = 0$. Let $\rho \in \pi_7 B_k$ project to $P(0, 1)$. In $E_2^{14,3}$, $h_0 P(0, 1)^2 = \Phi_0^2 \Phi_1 P(0, 1) \neq 0$. Note $E_2^{15,0} = E_2^{15,1} = 0$, $E_2^{15,2} = Z_2(P(0, 2))$, and $E_2^{15,3} = Z_2(h_0 P(0, 2))$. Observe that $P(0, 2) \in \langle \Phi_0, h_0, \Phi_2 \rangle$ in E_2 by [5, Theorem 4.3(1)]. Now Φ_0 and Φ_2 are infinite cycles because there are no elements whose product with h_0 is 0 in degrees 0 and 12. $\langle \Phi_0, 2, \Phi_2 \rangle$ is defined because Φ_0, Φ_2 is the only nonzero element of degree one, thirteen, respectively. By [4, Theorem 8.1], $P(0, 2)$ is an infinite cycle. Thus, $h_0 P(0, 1)^2$ is nonzero in $E_\infty^{14,5}$ and $2\rho^2 \neq 0$, contradicting the fact that $\pi_* B_2$ is a commutative graded ring. Therefore, B_2 cannot exist. □

Recall that $H_* \mathrm{MSp} = Z_2[\xi_n^4 | n \geq 1] \otimes \mathfrak{S}$, where $\mathfrak{S}$ is a polynomial algebra of $\mathfrak{A}^*$-primitive elements.

Lemma 5. *There is a map of spectra $f: B_3^{(24)} \to \mathrm{MSp}$ such that*

(a) $f | B_3^{(0)} = \mu: S \to \mathrm{MSp}$;
(b) $f_*(\xi) \equiv \xi$ *modulo the ideal spanned by $\overline{\mathfrak{S}}$ for all $\xi \in H_n B_3$ with $n < 24$.*

Proof. (a) We construct f on $B_3^{(q)}$ by induction on $q \geq 0$. Let $f | B_3^{(0)} = \mu$. Note that $H_* B_3$ is nonzero only in degrees divisible by eight. Assume that f has been defined on the $8(t-1)$-skeleton of B_3, $1 \leq t \leq 3$. By [3, Lemma

VI.3.2], the obstruction to extending f to the $8t$-skeleton of B_3 is an element of $H^*(B_3; \pi_{8t-1}\mathrm{MSp})$. However, the first nonzero element of $\pi_r\mathrm{MSp}$ in a degree congruent to $3 \bmod 4$ occurs when $r = 31$. Thus, we can extend f to the 24-skeleton of B_3.

(b) If $\xi \in H_n B_3$, $n \leq 23$, then $(1 \otimes \varepsilon)\psi f_*(\xi) = (1 \otimes \varepsilon)(1 \otimes f_*)\psi(\xi) = (1 \otimes \varepsilon)\psi(\xi) = \xi \otimes 1$. Kernel $(1 \otimes \varepsilon)\psi$ equals the ideal spanned by the $\mathfrak{A}^*$-primitive elements of positive degree of $H_*\mathrm{MSp} = Z_2[\xi_n^4 | n \geq 1] \otimes \mathfrak{S}$, which is $\overline{\mathfrak{S}}$. Thus, $f_*(\xi) \equiv \xi$ modulo the ideal spanned by $\overline{\mathfrak{S}}$. □

Theorem 6. *A ring spectrum* B_3 *does not exist.*

Proof. Observe that ${}_ME_1^{11,1} = Z_2(h_{22}) = {}_ME_\infty^{11,1}$ because ${}_ME_1^{12,0} = 0$. Thus, $E_2^{11,1} = Z_2(R)$. A straightforward calculation shows that ${}_ME_2^{10,k} = 0$ for $k \geq 3$, $k \neq 6$, and that ${}_ME_2^{10,6} = Z_2(h_{11}^2 h_{20}^4)$. Thus, the only possibility for a nonzero differential on R is $d_5(R) = h_1^2 Q$, where h_1, Q is represented by h_{11}, h_{20}^4, respectively, in the May spectral sequence. Let $f: B_3^{(24)} \to \mathrm{MSp}$ denote the map of Lemma 5. Then $f_*(h_1^2 Q) = \eta^2 q_0 \neq 0$ in $\pi_{10}\mathrm{MSp}$. Thus, R is an infinite cycle. Let $\lambda \in \pi_{11}B_3$ project to R. Note that $0 \neq h_{10}h_{22}^2 \in {}_ME_1^{22,3}$ is an infinite cycle. Since ${}_ME_1^{23,k} = 0$ for $0 \leq k \leq 2$, $0 \neq h_0 R^2 \in E_2^{22,3}$ and $E_2^{23,0} = E_2^{23,1} = 0$. Therefore, $h_0 R^2$ is a nonbounding infinite cycle. Thus, $2\lambda^2 \neq 0$, which contradicts the fact that $\pi_* B_3$ is a commutative graded ring. Therefore, B_3 cannot exist. □

Bibliography

1. J. F. Adams, *On the non-existence of elements of Hopf invariant one*, Ann. of Math. (2) **72** (1960), 20–104.
2. E. H. Brown and F. P. Peterson, *A spectrum whose Z_p cohomology is the algebra of reduced p-th powers*, Topology **5** (1966), 149–154.
3. S. T. Hu, *Homotopy theory*, Pure Appl. Math., vol. 8, Academic Press, New York, 1959.
4. S. O. Kochman, *A chain functor for bordism*, Trans. Amer. Math. Soc. **239** (1978), 167–196.
5. ——, *The symplectic cobordism ring*, I. Mem. Amer. Math. Soc., no. 228, Amer. Math. Soc., Providence, RI, 1980.
6. ——, *The symplectic Adams-Novikov spectral sequence for spheres*, submitted.
7. A. Liulevicius, *Notes on homotopy of Thom spectra*, Amer. J. Math. **86** (1964), 1–16.
8. J. P. May, *The cohomology of restricted Lie algebras and of Hopf algebras; application to the Steenrod algebra*, Ph.D. thesis, Princeton Univ., 1964.
9. ——, *The cohomology of restricted Lie algebras and of Hopf algebras*, J. Algebra **3** (1966), 123–146.
10. J. W. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Ann. of Math. **81** (1965), 211–264.

Department of Mathematics, York University, 4700 Keele Street, North York, Ontario M3J 1P3 Canada