NONEXISTENCE OF GENERALIZED SCATTERING RAYS
AND SINGULARITIES OF THE SCATTERING KERNEL
FOR GENERIC DOMAINS IN $\mathbb{R}^3$

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Abstract. It is proved for fixed unit vectors $\omega \neq \theta$ in $\mathbb{R}^3$ and generic bounded open domains $\Omega \subset \mathbb{R}^3$ that there do not exist generalized $(\omega, \theta)$-rays in $\Omega = \mathbb{R}^3 \setminus \mathcal{D}$ containing nontrivial geodesics on $\partial \Omega$. Consequently, for generic domains the sojourn times of reflecting $(\omega, \theta)$-rays completely describe the set of singularities of the scattering kernel $s(t, \theta, \omega)$.

1. Introduction

Let $\Omega$ be a closed domain in $\mathbb{R}^3$ with $C^\infty$ smooth boundary $\partial \Omega$ and bounded complement $\mathcal{D} = \mathbb{R}^3 \setminus \Omega$. For fixed $\omega, \theta \in S^2$ the scattering kernel $s(t, \theta, \omega) = s_{\Omega}(t, \theta, \omega)$ related to the wave equation in $\mathbb{R} \times \Omega$ with Dirichlet boundary conditions on $\mathbb{R} \times \partial \Omega$ is a distribution in $\mathcal{S}'(\mathbb{R})$ (see [6] for the definition). It was suggested by Guillemin [3] that the analysis of the singularities of $s(t, \theta, \omega)$ is connected with the sojourn times of the $(\omega, \theta)$-rays in $\Omega$.

Let $\gamma : \mathbb{R} \rightarrow \Omega$ be a generalized geodesic in $\Omega$; i.e. $\gamma = \iota \circ \hat{\gamma}$, where $\hat{\gamma} : \mathbb{R} \rightarrow T^*(\mathbb{R} \times \Omega)$ is a generalized bicharacteristic of the wave operator $\square = \partial_t^2 - \Delta$ (see [7] or [4, §24.3]) and $\iota : T^*(\mathbb{R} \times \Omega) \rightarrow \Omega$ is the canonical projection. If there exist real numbers $a < b$ such that $\hat{\gamma}(t) = \omega$ for $t \leq a$ and $\hat{\gamma}(t) = \theta$ for $t \geq b$, then $\gamma$ (and sometimes $\text{Im} \gamma$) is called a $(\omega, \theta)$-ray in $\Omega$. Such a curve $\gamma$ consists of linear segments in $\Omega$ (two of them are infinite straightline rays) and gliding segments (i.e. geodesics with respect to the standard metric) on $\partial \Omega$. If $\text{Im} \gamma$ contains only a finite number of linear segments and does not contain gliding ones, then $\gamma$ is called a reflecting $(\omega, \theta)$-ray in $\Omega$, otherwise $\gamma$ is called a generalized $(\omega, \theta)$-ray. By $\mathcal{L}_{\omega, \theta} = \mathcal{L}_{\omega, \theta}(\Omega)$ we denote the set of all $(\omega, \theta)$-rays in $\Omega$.

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Fix an open ball $B$ with radii $a > 0$ that contains $\mathcal{D}$. For $\eta \in S^2$ let $Z_\eta$ be the hyperplane in $\mathbb{R}^3$ tangent to $B$ such that $Z_\eta$ is orthogonal to $\eta$ and the halfspace $H_\eta$, determined by $Z_\eta$ and having $\eta$ as an inward normal, contains $B$. For a $(\omega, \theta)$-ray $\gamma$ in $\Omega$ denote by $T_\gamma$ the length of this part of $\gamma$ that is contained in $H_\omega \cap H_\eta$. Then $T_\gamma = T'_\gamma - 2a$ is called the sojourn time of $\gamma$ (cf. Guillemin [3]). It is easy to see that the definition of $T_\gamma$ does not depend on the choice of the ball $B$.

Under some assumptions on $\Omega$ it was established in [10] that
\begin{equation}
\text{sing supp}_s(t, \theta, \omega) \subset \{ -T_\gamma : \gamma \in \mathcal{L}_{\omega, \theta} \},
\end{equation}
where $\mathcal{L}_{\omega, \theta} = \mathcal{L}_{\omega, \theta}(\Omega)$ is the set of all $(\omega, \theta)$-rays in $\Omega$. Moreover, in [10] a formula was proved for the main singularity of $s(t, \theta, \omega)$ for $t$ close to some $T \in \text{sing supp}_s(t, \theta, \omega)$. Recently, the inclusion (1) was established in [1] under weaker assumptions, and it was also shown there that for generic $\Omega$ in $\mathbb{R}^n$, $T_\gamma \in \text{sing supp}_s(t, \theta, \omega)$ for a class of reflecting $(\omega, \theta)$-rays $\gamma$ in $\Omega$. In [8, 9, 14] all singularities of $s(t, \theta, \omega)$ have been examined for special classes of obstacles $\mathcal{D}$.

For $X = \partial \Omega$ denote by $C^{\infty}(X, \mathbb{R}^3)$ the space of all $C^{\infty}$ maps of $X$ into $\mathbb{R}^3$ endowed with the Whitney $C^{\infty}$ topology (cf. [2]), and by $C(X) = C^{\infty}_{\text{emb}}(X, \mathbb{R}^3)$ its open subset consisting of all $C^{\infty}$ embeddings of $X$ into $\mathbb{R}^3$. Then $C(X)$ is a Baire topological space, so every residual subset (i.e. a countable intersection of open dense subsets), is dense in it. Given $f \in C(X)$ we denote by $\Omega_f$ the unbounded closed domain in $\mathbb{R}^3$ with $\partial \Omega_f = f(X)$.

The main result in this paper is the following:

**Theorem 1.1.** Let $\theta \neq \omega$ be fixed unit vectors in $\mathbb{R}^3$. Then there exists a residual subset $\mathcal{R}$ of $C(X)$ such that for each $f \in \mathcal{R}$ there are no generalized $(\omega, \theta)$-rays in $\Omega_f$, and
\begin{equation}
\text{sing supp}_{s_{\Omega_f}}(t, \theta, \omega) = \{ -T_\gamma : \gamma \in \mathcal{L}_{\omega, \theta}(\Omega_f) \}.
\end{equation}

Moreover, for $f \in \mathcal{R}$ and $\gamma \in \mathcal{L}_{\omega, \theta}(\Omega_f)$ the main singularity of $s_{\Omega_f}(t, \theta, \omega)$ for $t$ near $-T_\gamma$ is given by the same formula as that in [10, 1] (see [1, Theorem 2]).

### 2. Degenerate $(\omega, \theta)$-rays

Let $\Omega$ be as in the introduction, $X = \partial \Omega$, and $\omega \in S^2$.

A curve $\gamma$ in $\Omega$ is called a degenerate $\omega$-ray if it has the form $\gamma = \bigcup_{i=0}^{k-1} l_i \subset \Omega$ and the following conditions are satisfied:

(i) $l_0$ is the infinite linear segment starting at $x_1$ having direction $-\omega$;
(ii) for every $i = 1, \ldots, k-1$, $l_i$ is a linear segment $[x_i, x_{i+1}]$, $x_i \in X$ for $i = 1, \ldots, k$;
(iii) if $k \geq 2$ then for any $i = 1, \ldots, k-2$, the segments $l_i$ and $l_{i+1}$ satisfy the reflection law at $x_{i+1}$ with respect to $X$; i.e., $l_i$ and $l_{i+1}$ make equal acute
angles with the interior (with respect to $\Omega$) unit normal vector $\nu(x_{i+1})$ to $X$ at $x_{i+1}$ and $l_i$, $l_{i+1}$, $\nu(x_{i+1})$ lie in a common plane; and

(iv) $l_{k-1}$ is tangent to $X$ at $x_k$, determining an asymptotic direction for $X$ at $x_k$ (cf. [15] for the definition of asymptotic direction).

The points $x_1, \ldots, x_k$ are called vertices of $\gamma$. If every segment of $\gamma$ is not tangent to $X$, then $\gamma$ is called ordinary. The defect of such a ray $\gamma$ is defined by $d(\gamma) = k - s$, where $s = \text{card}\{x_1, \ldots, x_k\}$.

Note that if the curvature of $X$ does not vanish of infinite order, then for every generalized $(\omega, \theta)$-ray $\gamma$ in $\Omega$ there exist a degenerate $\omega$-ray $\gamma_1$ and a degenerate $(-\theta)$-ray $\gamma_2$ with $\gamma_i \subset \text{Im} \gamma$, $i = 1, 2$ (cf. [7]).

For a set $A$ and an integer $s \geq 2$ we use the notation:

$$A^{(s)} = \{(a_1, \ldots, a_s) \in A^s : a_i \neq a_j \text{ for } i \neq j\}.$$ 

**Lemma 2.1.** There exists a residual subset $\mathcal{R}(\omega)$ of $C(X)$ such that for $f \in \mathcal{R}(\omega)$ if $\gamma$ is a degenerate $\omega$-ray in $\Omega_f$, then $d(\gamma) = 0$.

To prove the assertion one can use some arguments from the proof of Lemma 2.2 as well as a combinatorial classification of the degenerate $\omega$-rays, similar to that for periodic reflecting rays used in §4 of [11]. Since the modifications are rather standard, we omit the details.

For an integer $k \geq 1$ and $\omega \in S^2$, denote by $\mathcal{D}(\omega; k)$ the set of those $f \in C(X)$ such that the set of all $y = (y_1, \ldots, y_k) \in f(X)^{(k)}$ for which $y_1, \ldots, y_k$ are the successive vertices of a degenerate $\omega$-ray on $f(X)$ is a discrete subset of $f(X)^{(k)}$.

**Lemma 2.2.** The set $\mathcal{D}(\omega; k)$ contains a residual subset of $C(X)$.

**Proof.** To prove the assertion it is sufficient to establish that $\mathcal{D}(\omega; k) \cap C^\infty_{\text{emb}}(X, H_\omega)$ contains a residual subset of $C^\infty_{\text{emb}}(X, H_\omega)$.

We proceed as in [11-13]. Let $\pi: \mathbb{R}^3 \to \mathbb{Z} = \mathbb{Z}_\omega$ be the orthogonal projection. Denote by $U_k$ the set of those $y = (y_1, \ldots, y_k) \in (\mathbb{R}^3)^{(k)}$ such that for every $i = 1, \ldots, k-1$, $y_i$ does not belong to the segment $[y_{i-1}, y_{i+1}]$, where by definition $y_0 = \pi(y_1)$. Define $F: U_k \to \mathbb{R}$ by

$$F(y) = \sum_{i=0}^{k-1} ||y_i - y_{i+1}||.$$ 

If $y_1, \ldots, y_k$ are the successive reflection points of a degenerate $\omega$-ray $\gamma$ in $\Omega_f$ with $d(\gamma) = 0$, then $y = (y_1, \ldots, y_k) \in U_k$ and $F(y)$ is the length of $\gamma \cap H_\omega$. Moreover, for $y' = (y_1, \ldots, y_{k-1})$ we have $\nabla_y F(y) = 0$, $\langle y_k - y_{k-1}, \nu(y_k) \rangle = 0$, and $w = y_k - y_{k-1}$ is an asymptotic direction for $Y$ at $y_k$. The last condition can be expressed analytically as follows: Let $r: V \to Y$ be a chart, where $V$ is an open subset of $\mathbb{R}^2$ and $r(V)$ is an open neighborhood of $y_k$ in $Y$. Then $w = \lambda(\partial r/\partial u_1)(u) + \mu(\partial r/\partial u_2)(u)$ for some $\lambda, \mu \in \mathbb{R}$, where $r(u) = y_k$, $u = (u_1, u_2)$. Let $L$, $M$, $N$ be the coefficients of the
second fundamental form of \( Y \) at \( y_k \); that is, \( L(u) = \langle (\partial^2 r/\partial u_1^2)(u), \nu(y_k) \rangle \), 
\( M(u) = \langle (\partial^2 r/\partial u_1 \partial u_2)(u), \nu(y_k) \rangle \), 
\( N(u) = \langle (\partial^2 r/\partial u_2^2)(u), \nu(y_k) \rangle \). Then \( w \) is an asymptotic direction for \( Y \) at \( y_k \) iff (cf. [15])

\[
L(w, G\partial r/\partial u_1 - F\partial r/\partial u_2)^2 + 2M(w, G\partial r/\partial u_1 - F\partial r/\partial u_2)(w, E\partial r/\partial u_2 - F\partial r/\partial u_1) + N(w, E\partial r/\partial u_2 - F\partial r/\partial u_1)^2 = 0.
\]

It is easy to check that \( \lambda = \langle w, (G\partial r/\partial u_1 - F\partial r/\partial u_2)/(EG - F^2) \rangle \) and \( \mu = \langle w, (E\partial r/\partial u_2 - F\partial r/\partial u_1)/(EG - F^2) \rangle \), where \( E(u) = ||(\partial r/\partial u_1)(u)||^2 \), \( F(u) = ||(\partial r/\partial u_1)(u), (\partial r/\partial u_2(u)) \rangle \), and \( G(u) = ||(\partial r/\partial u_2)(u)||^2 \) are the coefficients of the first fundamental form. Therefore (3) is equivalent to

\[
L(w, G\partial r/\partial u_1 - F\partial r/\partial u_2)^2 + 2M(w, G\partial r/\partial u_1 - F\partial r/\partial u_2)(w, E\partial r/\partial u_2 - F\partial r/\partial u_1) + N(w, E\partial r/\partial u_2 - F\partial r/\partial u_1)^2 = 0.
\]

Let \( J^2_k(X, \mathbb{R}^3) \) be the \( k \)-fold bundle of 2-jets (cf. [2]). Given \( f \in C^\infty(X, \mathbb{R}^3) \), the map \( j^2_k f \colon X^{(k)} \to J^2_k(X, \mathbb{R}^3) \) is defined by \( j^2_k f(x_1, \ldots, x_k) = (j^2 f(x_1), \ldots, j^2 f(x_k)) \). Here \( j^2 f(x) \in J^2(X, \mathbb{R}^3) \) is the 2-jet determined by \( f \) at \( x \in X \). Denote by \( M \) the set of those \( \tau = (j^2 f_1(x_1), \ldots, j^2 f_k(x_k)) \in J^2_k(X, \mathbb{R}^3) \) such that \( (x_1, \ldots, x_k) \in X^{(k)} \); \( (f_1(x_1), \ldots, f_k(x_k)) \in U_k \); \( \text{rank} \, df_i(x_i) = 2 \) for every \( i = 1, \ldots, k \); and \( f_i(x_i) - f_{i+1}(x_{i+1}) \) is not tangent to \( f_i(X) \) at \( f_i(x_i) \) for all \( i = 1, \ldots, k-1 \), and \( \omega \) is not tangent to \( f_k(X) \) at \( f_k(x_k) \). Then \( M \) is open in \( J^2_k(X, \mathbb{R}^3) \). Finally, define the singularity set \( \Sigma \) as the set of those \( \tau \in M \) such that \( \text{grad}_x F \circ (f_1 \times \cdots \times f_k)(x) = 0 \), \( \langle f_k(x_k) - f_{k-1}(x_{k-1}), \nu \rangle = 0 \), and \( f_k(x_k) - f_{k-1}(x_{k-1}) \) is an asymptotic direction for \( f_k(X) \) at \( f_k(x_k) \), where \( \nu \) is a nonzero normal vector to \( f_k(X) \) at \( f_k(x_k) \).

Next, using some arguments from [11] or [12] (cf., for example, [11, proof of Lemma 7.1]) we establish that \( \Sigma \) is a smooth submanifold of \( M \) with codim \( \Sigma = 2k \). Then for \( f \in C^\infty(X, \mathbb{R}^3) \), \( j^2_k f \pitchfork \Sigma \) implies that \( \{x \in X^{(k)} : j^2_k f(x) \in \Sigma \} \) is a discrete subset of \( X^{(k)} \); i.e., \( f \in \mathcal{D}(\omega; k) \). Consequently \( \mathcal{D}(\omega; k) \) contains the residual subset:

\[
\{ f \in C^\infty(X, \mathbb{R}^3) : j^2_k f \pitchfork \Sigma \} \cap C^\infty_{\text{emb}}(X, H_\omega) \cap \mathcal{R}(\omega)
\]

of \( C^\infty_{\text{emb}}(X, H_\omega) \). This proves the assertion.

3. NONEXISTENCE OF GENERALIZED \((\omega, \theta)\)-RAYS

In this section we prove that generic domains \( \Omega \) do not admit generalized \((\omega, \theta)\)-rays. To this end we combine Lemma 2.2 with a simple perturbation technique.

**Lemma 3.1.** Let \( X \) be a smooth surface in \( \mathbb{R}^3 \) and \( c \colon [a, b] \to X \) be a geodesic on \( X(b > a) \). Let \( c(t_0) \) be an arbitrary point on the geodesic \((a < t_0 < b)\) that is not a point of selfintersection, and \( U \) be an arbitrary neighborhood of \( c(t_0) \).
in $X$ such that

$$\{c(t): t \in (\alpha, \beta)\} \cap \{a(t): a < t < \beta\}$$

for some $\alpha, \beta \in (a, b)$. Then there exists $f \in C(X)$ arbitrarily close to $id$ with respect to the $C^\infty$ topology such that $\text{supp} f \subset U$, and if $\hat{c}: [a, b] \to \hat{X}$ is the geodesic on $\hat{X} = f(X)$ with $\hat{c}(t) = c(t)$ for $t \in [a, \alpha)$, then

$$\{\hat{c}(t): t \in (\alpha, \beta]\} \cap \{c(t): t \in (\alpha, \beta]\} = \emptyset.$$  

**Proof.** We may assume that $U$ is small enough so that there exists coordinates $x_0, x_1$ in $U$ given by a chart $r: V \to U \subset X$, where $V = (\alpha, \beta) \times (-\delta, \delta) \subset \mathbb{R}^2$ for some $\delta > 0$, $a < \alpha < t_0 < \beta < b$, such that the components $g_{ij}$ of the standard metric $g$ on $X$ have the form:

$$g_{00}(x_0, x_1) = 1, \quad g_{01}(x_0, x_1) = 0, \quad g_{11}(x_0, x_1) = G(x_0, x_1) > 0$$

for $(x_0, x_1) \in V$. Moreover, we may assume

$$G(x_0, x_1) < 1 \quad \text{for all } (x_0, x_1) \in V.$$  

Otherwise we can replace $r$ by another chart, $\tilde{r}: V \to X$ given by $\tilde{r}(x_0, x_1) = r(x_0, ex_1)$; then $\tilde{g}_{11}(x_0, x_1) = e^2 g_{11}(x_0, x_1) < 1$ for $e > 0$ sufficiently small. Moreover, (4) holds provided $t_0 - \alpha, \beta - t_0, \text{ and } \delta$ are sufficiently small. Also note that $r(t, 0) = c(t)$ for $t \in (\alpha, \beta)$.

Take arbitrary $C^\infty$ functions $\lambda, \mu: \mathbb{R} \to [0, 1]$ with

$$\text{supp} \lambda = [\alpha, \beta], \quad p = p(0) > 0, \quad q = q'(0) > 0.$$  

For $e > 0$ small enough set $f_e(y) = y$ for $y \in X \setminus U$ and $f_e(y) = r(x, ex_1)$ for $y = r(x), x = (x_0, x_1) \in V$. Let $X_e = f_e(X)$. Then $\psi(x) = r(x) + e\lambda(x_0)\mu(x_1)(\partial r/\partial x_0)(x)$ defines a chart, $\psi: V \to \psi(V) \subset X_e$. Let $g_{ij}(e)$ be the standard metric on $X_e$ induced by $\mathbb{R}^3$; then for its components $g_{ij}(e; x_0, x_1)$ we have:

$$g_{00}(e; x_0, x_1) = 1 + 2e\lambda'(x_0)\mu(x_1) + O(e^2),$$

$$g_{01}(e; x_0, x_1) = e\lambda(x_0)\mu'(x_1) + O(e^2),$$

$$g_{11}(e; x_0, x_1) = G(x_0, x_1) + 2e\lambda(x_0)\mu(x_1)$$

$$\cdot (\partial r/\partial x_1)(x), (\partial^2 r/\partial x_0\partial x_1)(x)) + O(e^2)$$

for $e$ close to 0.

Using $(x_0, x_1)$ consider the canonical coordinates $x_0, x_1, y_0, y_1$ in $T^*X_e$ and the Hamiltonian vectorfield generated by the Hamiltonian:

$$H(e, x, y) = g_{00}(e; x)y_0^2/2 + g_{01}(e; x)y_0y_1 + g_{11}(e; x)y_1^2/2,$$

where $x = (x_0, x_1), y = (y_0, y_1)$. Let $c(e; t), 0 \leq t$, be the geodesic on $X_e$ with $c(e; t) = c(t)$ for each $t \in [0, \alpha]$, and $(x^{(e)}(t), y^{(e)}(t))$ be the corresponding integral curve in $T^*X_e$. Writing the Hamiltonian equations for this curve, and then the corresponding variational equations for

$$X_i(t) = (d/d\epsilon)x_i^{(e)}(t)|_{\epsilon=0}, \quad Y_i(t) = (d/d\epsilon)y_i^{(e)}(t)|_{\epsilon=0},$$
we get (cf. (7)):

\[ \begin{align*}
\dot{X}_0(t) &= Y_0(t) + 2p\lambda'(t) \\
\dot{X}_i(t) &= G(t, 0)Y_i(t) + q\lambda(t) \\
\dot{Y}_0(t) &= -p\lambda''(t) \\
\dot{Y}_i(t) &= -q\lambda'(t) \\
X_0(\alpha) &= X_1(\alpha) = Y_0(\alpha) = Y_1(\alpha) = 0
\end{align*} \]

for \((\alpha \leq t \leq \beta)\). Consequently, \(Y_1(t) = -q\lambda(t)\) and \(\dot{X}_1(t) = q\lambda(t)(1 - G(t, 0))\). Hence (6) yields \(\dot{X}_1(t) > 0\) for every \(t \in (\alpha, \beta)\), and therefore, \(X_1(t) > 0\) for each \(t \in (\alpha, \beta)\). This means that \((d/d\varepsilon)x^{(\varepsilon)}(t) > 0\) for \(t \in (\alpha, \beta)\), provided \(\varepsilon > 0\) is sufficiently small. Fix such an \(\varepsilon\). Then \(x^{(\varepsilon)}(t)\) is positive for \(t \in (\alpha, \beta)\), and for \(f = f_\varepsilon, \tilde{X} = X_\varepsilon, \) and \(\tilde{c}(t) = c(\varepsilon, t)\) we have (5). This proves the assertion.

Fix two unit vectors \(\omega \neq \theta\) in \(\mathbb{R}^3\).

**Theorem 3.2.** There exists a residual subset \(\mathcal{V}\) of \(C(X)\) such that for every \(f \in \mathcal{V}\) there are no generalized \((\omega, \theta)\)-rays in \(\Omega_f\).

**Proof.** We are going to construct by induction a decreasing sequence \(\mathcal{V}_1 \supset \mathcal{V}_2 \supset \cdots \supset \mathcal{V}_k \supset \cdots\) of residual subsets of \(C(X)\) such that for any \(k\) and any \(f \in \mathcal{V}_k\) there are no generalized \((\omega, \theta)\)-rays in \(\Omega_f\), with not more than \(k\) vertices.

It follows by [5] that there exists a residual subset \(\mathcal{H}\) of \(C(X)\) such that whenever \(f \in \mathcal{H}\), for every \(y \in f(X)\) the curvature of \(f(X)\) at \(y\) does not vanish of third order with respect to any direction tangent to \(f(X)\) at \(y\). Then for \(f \in \mathcal{H}\), if \(\gamma: \mathbb{R} \to \Omega_f\) is a \((\omega, \theta)\)-ray, then \(\text{Im} \gamma = \bigcup_{i=0}^{k} l_i\), where \(l_0\) and \(l_k\) are infinite segments, starting at \(x_1\) and \(x_k\), respectively, with directions \(-\omega\) and \(\theta\), and for any \(i = 1, \ldots, k - 1\), \(l_i\) is either a linear segment \([x_i, x_{i+1}]\) in \(\Omega_f\) or a geodesic \(x_i x_{i+1}\) on \(\partial \Omega_f = f(X)\), and \(x_i = \partial \Omega_f\) for every \(i = 1, \ldots, k\). Moreover, two successive linear segments of \(\gamma\) satisfy the reflection law at their common end, and if a linear and a gliding segments of \(\gamma\) are successive, then the linear segment is tangent to the gliding one, determining an asymptotic direction for \(\partial \Omega_f\) at their common end (cf. [7]).

Denote by \(\mathcal{V}_1\) the set of those \(f \in \mathcal{H}\) such that there are no degenerate \((\omega, \theta)\)-rays in \(\Omega_f\) with one vertex. The verification that \(\mathcal{V}_1\) is open and dense in \(C(X)\) uses some arguments very similar to those below, so we omit the details here.

Let \(k > 1\), and suppose we have already constructed the sets \(\mathcal{V}_1 \supset \cdots \supset \mathcal{V}_{k-1}\) so that they have the desired properties. Next, we construct \(\mathcal{V}_k\).

A function \(\mathcal{H}: \{1, 2, \ldots, k\} \to \{0, 1\}\) is called a \(k\)-design if \(\mathcal{H} \neq 0\), \(\mathcal{H}(0) = \mathcal{H}(k) = 0\), and \(\mathcal{H}(i) \mathcal{H}(i + 1) = 0\) for each \(i = 1, 2, \ldots, k - 2\). If \(\gamma\) is a generalized \((\omega, \theta)\)-ray with \(\text{Im} \gamma = \bigcup_{i=0}^{k} l_i\) and \(k\) vertices, and for each
i = 1, \ldots, k - 1, \mathcal{H}(i) = 0 \text{ holds iff } l_i \text{ is a linear segment, then } \gamma \text{ is called a ray with design } \mathcal{H}.

Fix a k-design \mathcal{H} and set

\[ q = \max\{i: 1 \leq i \leq k - 1, \mathcal{H}(i) = 1\}, \]

\[ p = \min\{i: 1 < i < k - 1, \mathcal{H}(i) = 1\}. \]

We now use the sets \mathcal{D}(\omega, m) defined in §2. Also consider the set \mathcal{I}_k of all \( f \in C(X) \) such that there are only finitely many reflecting (\omega, \theta)-rays in \Omega_f with not more than \( k \) vertices and all of them are ordinary. By Theorem 5.1 in [13], \mathcal{I}_k contains a residual subset of \( C(X) \). Then by Lemma 2.2 the set:

\[ \mathcal{W} = \mathcal{I}_k \cap \mathcal{D}(\omega, p) \cap \mathcal{D}(-\theta, q) \cap \mathcal{I}_k \]

also contains a residual subset of \( C(X) \). Fix an arbitrary \( r \in \mathbb{N} \) and denote by \( \mathcal{V}(k; r; \mathcal{H}) \) the set of those \( f \in \mathcal{W} \) such that there are no generalized (\omega, \theta)-rays in \( \Omega_f \) with design \mathcal{H} and sojourn times \( \leq r \).

First, we show that \( \mathcal{V}(k; r; \mathcal{H}) \) is dense in \( \mathcal{W} \). To this end we assume \( \text{id} \in \mathcal{W} \), and then we have to prove that there exists \( f \in \mathcal{V}(k; r; \mathcal{H}) \) arbitrarily close to \( \text{id} \) with respect to the \( C^\infty \) topology. Observe that there are only finitely many generalized (\omega, \theta)-rays in \( \Omega_f \) with design \mathcal{H} and sojourn times \( \leq r \).

Indeed, assume there exists an infinite sequence \{\gamma_m\} of distinct generalized (\omega, \theta)-rays \( \gamma_m : \mathbb{R} \to \Omega \) with design \mathcal{H} and sojourn times \( \leq r \). Let \( x_i^{(m)} = \gamma(t_i^{(m)}), \) \( 0 = t_1^{(m)} < t_2^{(m)} < \cdots < t_k^{(m)} \) be the successive vertices and \( l_i^{(m)} (i = 0, 1, \ldots, k) \) be the successive segments of \( \gamma_m \). We may assume that there exist \( \lim_m x_i^{(m)} = x_i \) for \( i = 1, \ldots, k \) and \( \lim_m |l_i^{(m)}| \) \( (|l| \text{ denotes the length of the segment } l) \) for \( i = 0, 1, \ldots, k \). Then a standard continuity argument shows that for every \( t \in \mathbb{R} \) there exists \( \lim_m \gamma_m(t) = \gamma(t) \), and \( \gamma \) is a (\omega, \theta)-ray in \( \Omega \) (cf. [4]). Moreover, \( l_i = \lim_m l_i^{(m)} \) are successive segments of \( \gamma \) (some of them may consists of only one point so they can be cancelled) with endpoint \( x_1, \ldots, x_k \) and \( \text{Im} \gamma = \bigcup_{i=0}^{k} l_i \). Since \( \theta \neq \omega \) and \( \text{id} \in \mathcal{W} \subset \mathcal{V}_1 \), the case \( x_1 = \cdots = x_k \) is impossible.

If every \( l_i \) is nontrivial, i.e. it does not consist of one point, then \( \gamma \) would be a generalized (\omega, \theta)-ray with design \mathcal{H}. Then \( \delta_m = \bigcup_{i=0}^{p-1} l_i^{(m)} \) for any \( m \in \mathbb{N} \) and \( \delta = \bigcup_{i=0}^{p-1} l_i \) are degenerate \( \omega \)-rays in \( \Omega \) with \( \delta_m \bigcap \delta = \emptyset \), which is a contradiction with \( \text{id} \in \mathcal{D}(\omega, \omega) \). Therefore \( l_i \) vanishes for at least one \( i \). It then follows that \( \gamma \) is a reflecting (\omega, \theta)-ray in \( \Omega \), otherwise we would get a contradiction with \( \text{id} \in \mathcal{V}_{k-1} \). Clearly, \( \gamma \) has at most \( k - 1 \) reflection points.

Moreover, applying some arguments similar to those in §4 of [13], we see that some segment of \( \gamma \) is tangent to \( X \), which is a contradiction with \( \text{id} \in \mathcal{I}_k \).

Hence there exist only finitely many generalized (\omega, \theta)-rays \( \gamma, \gamma_2, \ldots, \gamma_n \) in \( \Omega \) with design \mathcal{H} and sojourn times \( \leq r \). Let \( \text{Im} \gamma = \bigcup_{i=0}^{k} l_i \). Then \( l_j \) is a linear segment iff \( \mathcal{H}(j) = 0 \). Let \( x_i = \gamma(t_i) \) be the successive vertices of \( \gamma \),
Let \( 0 = t_1 < t_2 < \cdots < t_k \). Then

\[
I_q = \{ \gamma(t) : t_q \leq t \leq t_{q+1} \}
\]

is a geodesic on \( X \) and \( I_{q+1}, \ldots, I_{k-1} \) are linear segments. There is no \( a \in [t_q, t_{q+1}) \) such that

\[
\{ \gamma(t) : a \leq t \leq t_{q+1} \} \subset I_s
\]

for some \( s < q \). Indeed, if such \( a \) and \( s \) exist, then there would be two distinct generalized geodesics in \( \Omega \) passing through \( x_{q+1} \) in direction \( x_{q+1}x_{q+2} \), which is a contradiction with \( \text{id} \in \mathcal{W} \subset \mathcal{H} \) (cf. [7] or [4]). Hence for every \( s = 1, \ldots, q-1 \) there exists \( a \in [t_q, t_{q+1}) \) so that (9) does not hold. Consequently, there is \( t_0 \in (t_q, t_{q+1}) \) such that \( \gamma(t_0) \) is not a point of selfintersection of \( \gamma \). Moreover, applying the same argument, and eventually replacing \( V \) by a smaller neighborhood of \( x_k \), we see that \( t_0 \) can be chosen so that \( \gamma(t_0) \notin (\bigcup_{i=2}^n \text{Im} \gamma_i) \cup \mathcal{V} \). Next, choose a small coordinate neighborhood \( U \) of \( y(t_0) \) in \( X \) with

\[
U \cap \left( \bigcup_{i=2}^n \text{Im} \gamma_i \cup \mathcal{V} \cup \bigcup_{j=0}^{k} I_j \right) = \emptyset
\]

and such that (4) holds for \( c(t) = y(t), a = t_q, b = t_{q+1} \), and some \( \alpha, \beta \). By Lemma 3.1 there exists \( f \in C(X) \) arbitrarily close to \( \text{id} \) such that \( \text{supp } f \subset U \) and (5) holds for \( \bar{X} = f(X) \) and the geodesic \( \bar{c} : [a, b] \to \bar{X} \) with \( \bar{c}(t) = c(t) \) for \( t \in [a, \alpha] \). Since \( \text{id} \in \mathcal{D}(\omega; \rho) \) it is easily seen that if \( f \) is sufficiently close to \( \text{id} \), then the only generalized \((\omega, \theta)\)-rays in \( \Omega_f \) with design \( \mathcal{H} \) and sojourn times \( \leq r \) are \( \gamma_2, \ldots, \gamma_n \) and eventually, a ray \( \delta \) with \( \delta(0) = x_1 \) and \( \delta(0) = \omega \). Assume that for any choice of \( f \) there exists such a generalized \((\omega, \theta)\)-ray \( \delta = \delta_f \). Then clearly \( \delta(t) = \gamma(t) \) for all \( t \leq \alpha \). Let \( z_1 = x_1, z_2, \ldots, z_k \) be the successive vertices of \( \delta \). Observe that for \( f \) sufficiently close to \( \text{id} \) the last vertex \( z_k \) of \( \delta \) belongs to \( V \). Otherwise we would find a sequence \( f_m \to \text{id} \) such that the last vertex of \( \delta_m = \delta_{f_m} \) is not contained in \( V \) and \( \lim_m \delta_m(t) = \delta(t) \) exists for all \( t \in \mathbb{R} \); then \( \delta \) would be a generalized \((\omega, \theta)\)-ray in \( \Omega \) with design \( \mathcal{H} \) and sojourn time \( \leq r \) different from \( \gamma, \gamma_2, \ldots, \gamma_n \): a contradiction. Hence \( z_k \in V \) for \( f \) sufficiently close to \( \text{id} \), which is a contradiction with the choice of \( V \). Thus for \( f \) sufficiently close to \( \text{id} \) there are only \( n-1 \) generalized \((\omega, \theta)\)-rays with design \( \mathcal{H} \) and sojourn times \( \leq r \) in \( \Omega \). Moreover, a simple argument shows that for special construction of \( f \) considered above (if \( U \) is sufficiently small and \( f \) is sufficiently close to \( \text{id} \)) we have \( f \in \mathcal{W} \).

In this way, by induction we construct \( g \in \mathcal{V}(k; r; \mathcal{H}) \) arbitrarily close to \( \text{id} \). Hence \( \mathcal{V}(k; r; \mathcal{H}) \) is dense in \( \mathcal{W} \). To prove that \( \mathcal{V}(k; r; \mathcal{H}) \) is open in \( \mathcal{W} \) it is sufficient to establish that if \( \{ f_n \} \subset \mathcal{W} \setminus \mathcal{V}(k; r; \mathcal{H}) \) and \( f_n \to \text{id} \in \mathcal{W} \), then \( X \) admits a generalized \((\omega, \theta)\)-ray with design \( \mathcal{H} \) and
sojourn time \( \leq r \). This follows easily using some arguments from above and we omit the details.

The set \( \bigcap_{k \in \mathbb{N}} \mathcal{Y}(k, r; \mathcal{H}) \), where \( r \in \mathbb{N} \) and \( \mathcal{H} \) runs over the finite set of all \( k \)-designs, is a residual subset of \( \mathcal{W} \); therefore it contains a residual subset \( \mathcal{Y}_k \) of \( \mathcal{C}(X) \). Clearly \( \mathcal{Y}_k \) has the desired properties. This completes the construction of the sequence \( \{ \mathcal{Y}_k \} \).

Finally, setting \( \mathcal{V} = \bigcap_{k=1}^{\infty} \mathcal{Y}_k \), we obtain a residual subset of \( \mathcal{C}(X) \) such that for any \( f \in \mathcal{V} \) there are no generalized \((\omega, \theta)\)-rays in \( \Omega_f \).

**Proof of Theorem 1.1.** It follows by Theorem 2 in [1] that there exists a residual subset \( \mathcal{A} \) of \( \mathcal{C}(X) \) such that for any \( f \in \mathcal{A} \) and any reflecting \((\omega, \theta)\)-ray \( \gamma \) in \( \Omega_f \), if \( T_\gamma \neq T_\delta \) for every generalized \((\omega, \theta)\)-ray \( \delta \) in \( \Omega_f \), then \(-T_\gamma\) belongs to the left-hand side of (2). Set \( \mathcal{H} = \mathcal{A} \cap \mathcal{H} \cap \mathcal{V} \), then \( \mathcal{H} \) is a residual subset of \( \mathcal{C}(X) \). Given \( f \in \mathcal{H} \), by [10] or [1] we have that the left-hand side of (2) is contained in the right-hand side. Since \( f \in \mathcal{V} \), there are no generalized \((\omega, \theta)\)-rays in \( \Omega_f \), and the above remark implies that (2) holds.

**References**


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