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A NOTE ON THE PROPAGATORS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS IN HILBERT SPACES

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ABSTRACT. The paper is concerned with the growth properties at infinity of the propagators $C(\cdot)$, $S(\cdot)$ of the equation $u''(t) + Bu'(t) + Au(t) = 0$, where A, B are densely defined closed linear operators in a Hilbert space. We define $\omega_0(A, B) = \max\{\overline{\lim}_{t\to\infty} t^{-1} \ln \|C(t)\|, \overline{\lim}_{t\to\infty} t^{-1} \ln \|S'(t)\|\}$, and give a criterion to judge whether $\omega_0(A, B) \le b$ for a fixed $b \in R$.

Let H be a complex Hilbert space. We look at the second order linear differential equation

$$u''(t) + Bu'(t) + Au(t) = 0 \qquad (t \ge 0), \tag{1}$$

where A, B are densely defined closed linear operators in H. Recently, the authors [4] obtained a Hille–Yosida–Phillips type theorem for the Cauchy problem for (1) to be well posed. In the present paper we pay attention to the growth properties at infinity of the propagators of (1). Throughout this paper, notation and terminology is that of Fattorini [1] and the authors [4], and the Cauchy problem for (1) is assumed strongly well posed (see [4, Definition 4]). From [4] and [1, Chapter VIII], we know these facts: strong well posedness is equivalent to well posedness if $B \in L(H)$ (the set of bounded linear operators on H); any solution $u(\cdot)$ of (1) admits the representation $u(t) = C(t)u(0) + S(t)u'(0)$, with $u(0), u'(0) \in D(A) \cap D(B)$ (a dense set in H), where $C(\cdot)$, $S(\cdot)$ are the two propagators; there exist $C, b > 0$ such that for $\operatorname{Re}\lambda > b$, $\Delta(\lambda)^{-1} = (\lambda^2 I + \lambda B + A)^{-1} \in L(H)$ and for each $u \in H$, $t \ge 0$,

$$\begin{cases} \lambda\Delta(\lambda)^{-1}u = \int_0^\infty e^{-\lambda t} S'(t)u\,dt, & \|S'(t)\| \le Ce^{bt}, \\ \overline{\Delta(\lambda)^{-1}(\lambda + B_0)}u = \int_0^\infty e^{-\lambda t} C(t)u\,dt, & \|C(t)\| \le Ce^{bt}, \end{cases} \tag{2}$$

where $B_0 \subset B$ with $D(B_0) = D(A) \cap D(B)$; also,

$$S'(t) = C(t) - \overline{S(t)B_0}, \qquad t \ge 0. \tag{3}$$

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Now we define

$$\omega_0(A, B) = \max\left\{\overline{\lim_{t\to\infty}}\, t^{-1}\ln\|C(t)\|,\ \overline{\lim_{t\to\infty}}\, t^{-1}\ln\|S'(t)\|\right\}.$$

Clearly, $\omega_0(A, B) < \infty$. We note that for each $\varepsilon > 0$, there exists $C > 0$ such that $C(t)$, $S'(t)$ are bounded by $C\exp((\omega_0(A, B) + \varepsilon)t)$, while if they are bounded by $Ce^{\omega t}$ for some constants C, ω, then $\omega \geq \omega_0(A, B)$. Moreover, we see that $\overline{\lim}_{t\to\infty} t^{-1}\ln\|S(t)\| \leq \omega_0(A, B)$ from Theorem 1 below. Thus, when $B \in L(H)$, $\omega_0(A, B)$ coincides with $\max\{\overline{\lim}_{t\to\infty} t^{-1}\ln\|C(t)\|$, $\overline{\lim}_{t\to\infty} t^{-1}\ln\|S(t)\|\}$ due to (3).

Theorem 1. *Given* $\sigma_0 < b$ (b *is the constant in* (2)), $\omega_0(A, B) \leq \sigma_0$ *if and only if*

(i) *For each* $\operatorname{Re}\lambda > \sigma_0$, $\Delta(\lambda)^{-1}$ *exists and belongs to* $L(H)$, *and* $\Delta(\lambda)^{-1}B_0$ *has bounded extension* $\overline{\Delta(\lambda)^{-1}B_0}$.

(ii) $\operatorname{Sup}\{\max(\|\lambda\Delta(\lambda)^{-1}\|, \|\overline{\Delta(\lambda)^{-1}B_0}\|);\ \operatorname{Re}\lambda \geq \sigma\} < \infty$ *whenever* $\sigma > \sigma_0$.

Furthermore, in this case, $\overline{\lim}_{t\to\infty} t^{-1}\ln\|S(t)\| \leq \sigma_0$.

Proof. Necessity. For each $b > \sigma > \sigma_0$ (we may and do assume $\sigma \neq 0$), there exists by assumption, a constant $C \geq 1$ such that $\|C(t)\| \leq Ce^{\sigma_1 t}$, $\|S'(t)\| \leq Ce^{\sigma_1 t}$ for $t \geq 0$, where $\sigma_1 = \frac{1}{2}(\sigma_0 + \sigma)$. Hence from (2), we obtain that for $\operatorname{Re}\lambda \geq a > b$,

$$\|\lambda\Delta(\lambda)^{-1}\| \leq C(\operatorname{Re}\lambda - \sigma_1)^{-1},\ \|\overline{\Delta(\lambda)^{-1}(\lambda + B_0)}\| \leq C(\operatorname{Re}\lambda - \sigma_1)^{-1}. \tag{4}$$

Write $M = \max(8C(\sigma - \sigma_1)^{-1}, |\sigma|^{-1}, a^{-1})$. We now choose $\{\tau_k\}_{0\leq k\leq n}$ such that $\tau_0 = \sigma < \tau_1 < \cdots < \tau_{n-1} < \tau_n = a$, $\tau_k - \tau_{k-1} \leq M^{-1}$ $(1 \leq k \leq n)$, and if $\sigma < 0$, $\tau_{k_0 - 1} = -(2M)^{-1}$, $\tau_{k_0} = (2M)^{-1}$ for some k_0 $(1 \leq k_0 \leq n)$. Observe that for $\lambda = \tau + i\omega$, $\tau \in [\tau_{n-1}, \tau_n]$, $\omega \in R$, taking $\lambda_0 = \tau_n + i\omega$, then

$$\begin{aligned}&\|(\lambda - \lambda_0)[\lambda_0\Delta(\lambda_0)^{-1}(1 + \lambda_0^{-1}(\lambda - \lambda_0)) + \overline{\Delta(\lambda_0)^{-1}(\lambda_0 + B_0)}]\|\\ &\quad\leq M^{-1}[C(\sigma - \sigma_1)^{-1}(1 + 2MM^{-1} + 1)] \leq \tfrac{1}{2} < 1.\end{aligned}$$

whence $\Delta(\lambda)$ is invertible and

$$\Delta(\lambda)^{-1} = \{I + (\lambda - \lambda_0)[\lambda\Delta(\lambda_0)^{-1} + \overline{\Delta(\lambda_0)^{-1}(\lambda_0 + B_0)}]\}^{-1}\Delta(\lambda_0)^{-1} \in L(H);$$

the identity also indicates that $\Delta(\lambda)^{-1}B_0$ is closable and $\overline{\Delta(\lambda)^{-1}B_0} \in L(H)$. From these facts, it follows immediately that (2) remains valid for each $\operatorname{Re}\lambda \geq \tau_{n-1}$, and therefore, so does (4). Iterating this process a finite number of times yields the desired result.

Before justifying the sufficiency, we present the following elementary result.

Proposition 1. *Let* $f(t)$ $(t \geq 0)$ *be a strongly measurable function taking values in* H *that satisfies* $\|f(t)\| \leq Ce^{ct}$ *in* $t \geq 0$, *where* $C, c \geq 0$. *Then for each*

$\tau > c$, $\|L(\tau + i\omega)\| = \|\int_0^\infty e^{-(\tau+i\omega)t} f(t)\,dt\| \in L_2(R)$ *as a function of* $\omega \in R$; *also* $\|L(\tau + i\omega)\| \to 0$ *as* $|\omega| \to \infty$.

Proof of the sufficiency. Fix σ $(b > \sigma > \sigma_0)$. We first examine $[\lambda\Delta(\lambda)^{-1}]^*$ (the adjoint) and $[\overline{\Delta(\lambda)^{-1}B_0}]^*$ for $\operatorname{Re}\lambda > b$. Plainly, for each $u \in H$, $[S'(t)]^*u$ and $[C(t)]^*u$ are weakly continuous in $t \geq 0$. So they are weakly measurable. Moreover, by [3, Theorem 3.5, p. 59], we have that $[S'(t)]^*u \in \overline{\operatorname{span} X(u)}$ (resp. $[C(t)]^*u \in \overline{\operatorname{span} Y(u)}$) for each $t \geq 0$, where $X(u) = \{[S'(r)]^*u : r$ rational and in $[0, \infty)\}$ (resp. $Y(u) = \{[C(r)]^*u : r$ rational and in $[0, \infty)\}$). Therefore, $[S'(t)]^*u$ and $[C(t)]^*u$ are strongly measurable for each $u \in H$ by Pettis' theorem [2, Theorem 3.5.3, p. 72]. Now observing that for each $u, v \in H$, $\operatorname{Re}\lambda > b$,

$$\begin{aligned}\left(\int_0^\infty e^{-\lambda t} S'(t)u\,dt, v\right) &= \int_0^\infty (e^{-\lambda t} S'(t)u, v)\,dt \\ &= \int_0^\infty (u, e^{-\bar{\lambda} t}[S'(t)]^* v)\,dt \\ &= \left(u, \int_0^\infty e^{-\bar{\lambda} t}[S'(t)]^* v\,dt\right),\end{aligned}$$

then (2) shows that for each $u \in H$, $t \geq 0$,

$$[\lambda\Delta(\lambda)^{-1}]^* u = \int_0^\infty e^{-\bar{\lambda} t}[S'(t)]^* u\,dt, \qquad \|[S'(t)]^*\| = \|S'(t)\| \leq Ce^{bt}.$$

Likewise, we have that for each $u \in H$, $t \geq 0$,

$$[\overline{\Delta(\lambda)^{-1}B_0}]^* u = \int_0^\infty e^{-\bar{\lambda} t}[C(t) - S'(t)]^* u\,dt, \qquad \|[C(t) - S'(t)]^*\| \leq 2Ce^{bt}.$$

It follows from these facts and (2) that the conclusion of Proposition 1 hold with $L(\tau + i\omega)$ respectively replaced by $\lambda\Delta(\lambda)^{-1}u$, $\overline{\Delta(\lambda)^{-1}B_0}u$, $[\lambda\Delta(\lambda)^{-1}]^*u$, $[\overline{\Delta(\lambda)^{-1}B_0}]^*u$ for each $\operatorname{Re}\lambda > b$, $u \in H$. We assert that the range $\operatorname{Re}\lambda > b$ can be extended to $\operatorname{Re}\lambda \geq \sigma$; moreover, the limit is uniform on $[\sigma, a]$ $(a > b)$ as $|\omega| \to \infty$ for each one. To see why, set a partition of $[\sigma, a] : \sigma = \tau_0 < \cdots < \tau_n = a$ satisfying $\tau_k - \tau_{k-1} < (8C_0)^{-1}$

$(C_0 = \operatorname{Sup}\{\max(1, \|\lambda\Delta(\lambda)^{-1}\|, \|\Delta(\lambda)^{-1}\|, \|\overline{\Delta(\lambda)^{-1}B_0}\|); \operatorname{Re}\lambda \geq \sigma\})$, $1 \leq k \leq n$.

Then

$$\begin{aligned}\Delta(\lambda)^{-1} &= \{I + (\lambda - \lambda_0)[\lambda\Delta(\lambda_0)^{-1} + \overline{\Delta(\lambda_0)^{-1}(\lambda_0 + B_0)}]\}^{-1}\Delta(\lambda_0)^{-1} \\ &= (I + M_\lambda)^{-1}\Delta(\lambda_0)^{-1}\end{aligned}$$

(here $M_\lambda = (\lambda - \lambda_0)[\lambda\Delta(\lambda_0)^{-1} + \overline{\Delta(\lambda_0)^{-1}(\lambda_0 + B_0)}]$) $\|M_\lambda\| \leq \frac{1}{2}$, $\|(I + M_\lambda)^{-1}\| \leq 2$ valid for $\operatorname{Re}\lambda \in [\tau_{k-1}, \tau_k]$, $\lambda_0 = \tau_k + i(\operatorname{Im}\lambda)$, $1 \leq k \leq n$. The above assertion now follows from the equalities below:

$$\begin{aligned}
\lambda\Delta(\lambda)^{-1}u &= (\lambda-\lambda_0)(I+M_\lambda)^{-1}\Delta(\lambda_0)^{-1}u + (I+M_\lambda)^{-1}\lambda_0\Delta(\lambda_0)^{-1}u;\\
\overline{\Delta(\lambda)^{-1}B_0}u &= (I+M_\lambda)^{-1}\overline{\Delta(\lambda_0)^{-1}B_0}u;\\
[\lambda\Delta(\lambda)^{-1}]^*u &= [\bar\lambda-\bar\lambda_0+\bar\lambda_0][\Delta(\lambda_0)^{-1}]^*(I+M_\lambda^*)^{-1}(I+M_\lambda^*-M_\lambda^*)u\\
&= (\bar\lambda-\bar\lambda_0)\{[\Delta(\lambda_0)^{-1}]^*u-[\Delta(\lambda_0)^{-1}]^*(I+M_\lambda^*)^{-1}M_\lambda^*u\}\\
&\quad + [\lambda_0\Delta(\lambda_0)^{-1}]^*u-[\lambda_0\Delta(\lambda_0)^{-1}]^*(I+M_\lambda^*)^{-1}M_\lambda^*u;\\
[\overline{\Delta(\lambda)^{-1}B_0}]^*u &= [\overline{\Delta(\lambda_0)^{-1}B_0}]^*u-[\overline{\Delta(\lambda_0)^{-1}B_0}]^*(I+M_\lambda^*)^{-1}M_\lambda^*u,
\end{aligned}$$

for $u\in H$, $\operatorname{Re}\lambda\in[\tau_{k-1},\tau_k]$, $\lambda_0=\tau_k+i(\operatorname{Im}\lambda)$, $1\le k\le n$; as an example, $\|[\overline{\Delta(\lambda)^{-1}B_0}]^*u\|\in L_2(R)$ because of $\|[\overline{\Delta(\lambda)^{-1}B_0}]^*u\|\le 3C_0(\|[\Delta(\lambda_0)^{-1}]^*u\| + 2\|[\lambda_0\Delta(\lambda_0)^{-1}]^*u\|+\|[\overline{\Delta(\lambda_0)^{-1}B_0}]^*u\|)$.

Next, by (2), for $\operatorname{Re}\lambda>b$, $u\in H$,

$$\frac{1}{\lambda}\overline{\Delta(\lambda)^{-1}(\lambda+B_0)}u=\int_0^\infty e^{-\lambda t}\left(\int_0^t C(s)u\,ds\right)dt.$$

This together with [4, Lemma 2] gives that for $u, v\in H$, $t\ge 1$,

$$\left(\int_0^t C(s)u\,ds, v\right)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}e^{\lambda t}\left(\frac{1}{\lambda}\overline{\Delta(\lambda)^{-1}(\lambda+B_0)}u, v\right)d\lambda.$$

Integrating by parts, we obtain

$$\left(\int_0^t C(s)u\,ds, v\right)=-\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{\lambda t}}{t}\frac{d}{d\lambda}\left(\frac{1}{\lambda}\overline{\Delta(\lambda)^{-1}(\lambda+B_0)}u, v\right)d\lambda.$$

But

$$\begin{aligned}
&\left|\lambda\frac{d}{d\lambda}\left(\frac{1}{\lambda}\overline{\Delta(\lambda)^{-1}(\lambda+B_0)}u, v\right)\right|\\
&\quad=\left|(\Delta(\lambda)^{-1}u, v)-\left(\frac{1}{\lambda}\overline{\Delta(\lambda)^{-1}(\lambda+B_0)}u, v\right)\right.\\
&\qquad\left.-(\overline{\Delta(\lambda)^{-1}(2\lambda+B_0)\Delta(\lambda)^{-1}(\lambda+B_0)}u, v)\right|\\
&\quad=\left|\left(\lambda\Delta(\lambda)^{-1}u, \frac{1}{\bar\lambda}v\right)-\left(\overline{\Delta(\lambda)^{-1}(\lambda+B_0)}u, \frac{1}{\bar\lambda}v\right)\right.\\
&\qquad\left.-(\overline{\Delta(\lambda)^{-1}(\lambda+B_0)}u, [\overline{\Delta(\lambda)^{-1}(2\lambda+B_0)}]^*v)\right|\\
&\quad\in L_1(R).
\end{aligned}$$

The dominated convergence theorem then shows

$$(C(t)u, v)=-\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left(\frac{e^{\lambda t}}{t}-\frac{e^{\lambda t}}{\lambda t^2}\right)\lambda\frac{d}{d\lambda}\left(\frac{1}{\lambda}\overline{\Delta(\lambda)^{-1}(\lambda+B_0)}u, v\right)d\lambda,$$

which, integrating by parts again, is

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{\lambda t}\overline{(\Delta(\lambda)^{-1}(\lambda+B_0)u,\, v)}\, d\lambda \text{ whenever } u, v \in H.$$

Shifting the path of integration by virtue of analyticity, we have

$$\begin{aligned}|(C(t)u, v)| &= \left|\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t}\overline{(\Delta(\lambda)^{-1}(\lambda+B_0)u,\, v)}\, d\lambda\right| \\ &= \left|\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{e^{\lambda t}}{t}\left\{\left(\lambda\Delta(\lambda)^{-1}u,\, \frac{1}{\bar\lambda}v\right)\right.\right. \\ &\qquad \left.\left. - \overline{(\Delta(\lambda)^{-1}(\lambda+B_0)u,\, [\Delta(\lambda)^{-1}(2\lambda+B_0)]^{*}v)}\right\} d\lambda\right| \\ &\le e^{\sigma t}C(u, v)\end{aligned}$$

for some $C(u, v)$ depending uniquely on u, v. According to [3, Theorem 3.18] and the Banach–Steinhaus theorem, we conclude

$$\|C(t)\| \le Me^{\sigma t} \qquad (t \ge 0) \text{ for some } M > 0.$$

Similarly, we obtain the other two required estimates for $S(t)$, $S'(t)$. Since σ is arbitrary, the proof is thus complete.

Remark. Theorem 1 cannot be obtained directly from semigroup theory.

Corollary. *Suppose* (i) *and* (ii) *in Theorem* 1 *are satisfied for some* $\sigma_0 < 0$. *Then* $C(\cdot)$, $S(\cdot)$, *and* $S'(\cdot)$ *are all exponentially stable, i.e., there exist* $C > 0$, $\sigma < 0$ *such that they are bounded by* $Ce^{\sigma t}$.

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