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# ON INTERMEDIATE DIFFERENTIABILITY OF LIPSCHITZ FUNCTIONS ON CERTAIN BANACH SPACES

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ABSTRACT. A real-valued function $f$ defined on a Banach space $X$ is said to be intermediately differentiable at $x \in X$ if there is $\xi \in X^*$ such that for every $h \in X$ the value $\langle \xi, h \rangle$ lies between the upper and lower derivatives of $f$ at $x$ in the direction $h$. We show that if $Y$ contains a dense continuous linear image of an Asplund space and $X$ is a subspace of $Y$, then every locally Lipschitz function on $X$ is generically intermediately differentiable.

Let $(X, \|\cdot\|)$ be a Banach space with dual $X^*$ and duality pairing between $X^*$ and $X$ denoted by $\langle \cdot, \cdot \rangle$. Recall that the *upper* and *lower derivatives* of a function $f: X \to \mathbb{R}$ at $x \in X$ in a direction $h \in X$ are defined by

$$D^+ f(x, h) = \limsup_{t \downarrow 0} \tfrac{1}{t}[f(x + th) - f(x)]$$

and

$$D_+ f(x, h) = \liminf_{t \downarrow 0} \tfrac{1}{t}[f(x, th) - f(x)]$$

respectively. The function $f$ is said to be *intermediatly differentiable* at $x \in X$, with *intermediate derivative* $\xi \in X^*$, if

$$D^+ f(x, h) \geq \langle \xi, h \rangle \geq D_+(x, h) \quad \text{for all } h \in X.$$

The aim of this note is to prove the following statement.

**Theorem.** *Suppose that a Banach space $Y$ contains a dense continuous linear image of an Asplund space and that $X$ is a subspace of $Y$.*

*Then every locally Lipschitz function defined on an open subset $\Omega$ of $X$ is intermediately differentiable at every point of a residual subset of $\Omega$.*

The Banach spaces containing a dense continuous linear image of an Asplund space have been extensively studied by Ch. Stegall [11], who calls them *GSG spaces*. Among other things, he also proved [11, §4, Remark 3] that the non-w.c.g. subspace of some $L_1(\mu)$, $\mu$ finite, constructed by H. Rosenthal [10] is

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not GSG. Hence our theorem extends the result of M. Fabian and N. V. Zhivkov [5] who considered the case $X = Y$.

We observe that our result implies the well-known fact [2] that *a subspace $X$ of a GSG space is weak Asplund*. Indeed, using [15, Proposition 2.1], one easily sees that a stronger statement holds: *If a continuous function $f$ defined on an open subset $\Omega$ of $X$ has (one-sided) directional derivative at each point of $\Omega$ and at each direction, then $f$ is generically Gateaux differentiable on $\Omega$.*

It does not seem that the proof of the above theorem can be obtained by adapting the proofs from [5, 15]. In fact, a quite different approach works; we use some ideas met in the theory of analytic spaces on one hand and some two-dimensional reasoning similar to those used in [8, 9] on other hand.

The crucial concept we use is *$d$-$A$-dentability*$^*$. Let $E$ be a bounded subset of the dual $X^*$ of a Banach space $X$, let $A$ be a bounded absolutely convex subset of $X$; and let $d > 0$ be given. We say that $E$ is $d$-$A$-dentable$^*$ if there are $0 \neq e \in X$ and $\gamma > 0$ such that the slice

$$S(E, e, \gamma) \equiv \{\xi \in E : \langle \xi, e \rangle > \sup\langle E, e \rangle - \gamma\}$$

has $A$-diameter less than $d$; that is,

$$\operatorname{diam}_A S(E, e, \gamma) \equiv \sup\{\langle \xi - \eta, u \rangle : \xi, \eta \in S(E, e, \gamma),\ u \in A\} < d.$$

Let us recall that $A \subset X$ is said to have a *Stegall property* provided that every bounded subset of $X^*$ is $d$-$A$-dentable$^*$ for every $d > 0$; see [1, pp. 116–148], where other equivalent characterizations and other names of this property can also be found.

We also use the notion of a *Clarke subdifferential* [3]: Given a locally Lipschitz function $f : X \to \mathbb{R}$ on a Banach space $X$, put

$$f^0(x)(h) = \limsup_{\substack{t \downarrow 0 \\ y \to x}} \tfrac{1}{t}[f(y + th) - f(y)], \qquad x, h \in X.$$

Then the Clarke subdifferential of $f$ at $x \in X$ is defined by

$$\partial f(x) = \{\xi \in X^* : \langle \xi, h \rangle \le f^0(x)(h) \text{ for all } h \in X\}.$$

Note that $\partial f(x)$ is a nonempty set and that the function $f^0(x)(\cdot)$ is subadditive and positively homogeneous for each $x \in X$.

The key lemma we use can be formulated as:

**Lemma.** *Let $G$ be an open subset of a Banach space $X$, consider a Lipschitz function $f : G \to \mathbb{R}$ with a Lipschitz constant $1$, and put $\partial f(G) = \bigcup\{\partial f(x) : x \in G\}$. Let $A$ be an absolutely convex bounded subset of $X$ with the Minkowski's functional $q$. Finally, let $d > 0$, $\gamma > 0$, and $e \in X$, with $\|e\| = 1$, be such that*

$$\operatorname{diam}_A S(\partial f(G), e, \gamma) < d. \tag{1}$$

*Then there exist* $x$ *in* $G$, $\xi$ *in the unit ball* $B_{X^*}$ *of* $X^*$ *and* $\Delta > 0$ *such that*

$$|f(x+h) - f(x) - \langle \xi , h \rangle| \leq 3dq(h)$$

*whenever* $h \in X$ *and* $q(h) < \Delta$.

*Proof*. Put $\alpha = \sup\{\|u\| : u \in A\}$ and

$$s = \sup\langle \partial f(G) , e \rangle \tag{2}$$

Choose

$$\beta \in \left(0, \min\left(\gamma, \frac{d^2\gamma}{4\alpha(d+2\alpha)}\right)\right). \tag{3}$$

From the definition of $\partial f(G)$ and using classical facts from real analysis, we can find $x \in G$ such that the limit

$$Vf(x)(e) \equiv \lim_{t\to 0} \tfrac{1}{t}[f(x+te) - f(x)]$$

exists and $Vf(x)(e) > s-\beta$. In fact, find $y \in G$ such that $s-\beta < \sup\langle \partial f(y), e\rangle$; then, according to the definition of $\partial f(y)$, we get $s-\beta < f^0(y)(e)$. Also, by the definition of $f^0(y)(e)$, find $z \in G$ and $\tau > 0$ so that $\{z+te: t \in [0, \tau]\} \subset G$ and $s - \beta < \frac{1}{\tau}[f(z+\tau e) - f(z)]$. Now the mapping $t \mapsto g(t) \equiv f(z+te)$, $t \in [0, \tau]$, is Lipschitz. Hence the derivative $g'(t)$ exists almost everywhere and moreover,

$$\int_0^\tau g'(t)dt = f(z+\tau e) - f(z),$$

which is greater that $\tau(s-\beta)$. It follows that there is $t \in (0, \tau)$ such that $g'(t)$ exists and $g'(t) > s - \beta$. Then put $x = z + te$.

Further, from the definition of $\partial f(x)$, we can find $\xi \in \partial f(x)$ such that

$$Vf(x)(e) = \langle \xi , e \rangle \qquad ( > s - \beta). \tag{4}$$

Note that $\xi \in S(\partial f(G), e, \gamma)$ since $\beta < \gamma$. Also there is $\delta \in (0, 1)$ such that $u \in G$ whenever $\|u - x\| < 2\delta$ and such that

$$|f(x+\tau e) - f(x) - \tau\langle \xi , e \rangle| \leq \beta|\tau| \quad \text{for } \tau \in (-\delta, \delta). \tag{5}$$

Now fix any $h \in X$ such that $0 < q(h) < \min(\delta/\alpha, 2\beta\delta/d)$, and take

$$t = dq(h)/2\beta. \tag{6}$$

Then note that $\|h\| \leq \alpha q(h) < \delta$ and $0 < t < (d/2\beta)\cdot(2\beta\delta/d) = \delta$. Let us remark that for $(\tau, \rho) \in [0, t] \times [0, 1] \equiv I$ we have

$$\|x + \tau e + \rho h - x\| \leq \tau + \rho\|h\| \leq t + \|h\| < 2\delta;$$

so $x + \tau e + \rho h \in G$. Thus for $\rho \in [0, 1]$, we get

$$\begin{aligned} f(x+te+\rho h) &\geq f(x+te) - \rho\|h\| \\ &\geq f(x) + t\langle \xi , e\rangle - \beta t - \rho\|h\| \\ &> f(x) + ts - 2\beta t - \|h\| \end{aligned}$$

by (5) and (4); and further

$$f(x+\rho h) \le f(x) + \|h\|.$$

And putting the last two inequalities together with $\|h\| \le \alpha q(h)$, (6), and (3), we get

$$\begin{aligned} f(x+te+\rho h) - f(x+\rho h) &> ts - 2\beta t - 2\|h\| \\ &\ge ts - 2\beta t - 2\alpha q(h) \\ &= ts - 2\beta t(1+2\alpha/d) > ts - d\gamma t/2\alpha, \end{aligned}$$

(7) $$f(x+te+\rho h) - f(x+\rho h) > ts - d\gamma t/(2\alpha) \quad \text{for all } \rho \in [0, 1].$$

We now define a mapping $\phi : I \to \mathbb{R}$ by

$$\phi(\tau, \rho) = f(x+\tau e+\rho h), \qquad (\tau, \rho) \in I.$$

Note that $\phi$ is Lipschitz and so by the Rademacher theorem, there is a Lebesgue negligible set $N$ in $I$ such that the Gateaux (i.e. Fréchet) derivative $\phi'(\tau, \rho)$ exists whenever $(\tau, \rho) \in I\backslash N$. Let us observe that for these $(\tau, \rho)$

$$\begin{aligned} \phi_1'(\tau, \rho) &= Vf(x+\tau e+\rho h)(e) \le s, \\ \phi_2'(\tau, \rho) &= Vf(x+\tau e+\rho h)(h) \ge -\|h\| \ge -\alpha q(h), \end{aligned}$$

where $\phi_1'$ and $\phi_2'$ mean the partial derivatives of $\phi$ along the first and the second coordinate respectively. Let $\lambda_1$ and $\lambda_2$ denote the one- and the two-dimensional Lebesgue measures respectively.

Consider the following sets:

(8) $$\begin{aligned} E &= \{(\tau, \rho) \in I\backslash N : \phi_1'(\tau, \rho) > s - \gamma\}; \\ E_\tau &= \{\rho \in [0, 1] : (\tau, \rho) \in E\}, \qquad \tau \in [0, t]; \\ E^\rho &= \{\tau \in [0, t] : (\tau, \rho) \in E\}, \qquad \rho \in [0, 1]. \end{aligned}$$

Using (7) and (8) we can estimate for almost all $\rho \in [0, 1]$,

$$\begin{aligned} ts - \frac{d\gamma t}{2\alpha} < \phi(t, \rho) - \phi(0, \rho) &= \int_0^t \phi_1'(\tau, \rho) d\tau = \int_{E^\rho} + \int_{E^{\rho c}} \\ &\le \lambda_1(E^\rho)s + (t - \lambda_1(E^\rho))(s-\gamma) = \lambda_1(E^\rho)\gamma + ts - t\gamma; \end{aligned}$$

$$\lambda_1(E^\rho) > t - dt/(2\alpha):$$

$$\Lambda_2(e) = \int_0^1 \lambda_1(e^\rho) d\rho > t - dt/(2\alpha).$$

On the other hand, as $\lambda_2(E) = \int_0^t \lambda_1(E_\tau) d\tau$, we can find $\tau \in [0, t]$ such that

(9) $$\lambda_1(E_\tau) > 1 - d/(2\alpha)$$

and that, simultaneously, the derivative $\phi'(\tau, \rho)$ exists for almost all $\rho \in [0, 1]$. For every such $\rho$ we have $(\tau, \rho) \notin N$ and by the Hahn-Banach theorem, there exists $\eta_\rho \in \partial f(x+\tau e+\rho h)$ such that

$$\langle \eta_\rho, u \rangle = Vf(x+\tau e+\rho h)(u)$$

for all $u$ from the linear span of $e$ and $h$. Therefore, observing that, by (8), $\eta_\rho \in S(\partial f(G), e, \gamma)$ for $\rho \in E_\tau$, we have by (1) and (9)

$$\begin{aligned}|f(x+\tau e+h) - f(x+\tau e) - \langle \xi, h\rangle| &= \left|\int_0^1 \phi_2'(\tau, \rho)d\rho - \langle \xi, h\rangle\right| \\ &= \left|\int_0^1 \langle \eta_\rho - \xi, h\rangle d\rho\right| \le \left|\int_{E_\tau}\right| + \left|\int_{E_\tau^c}\right| \\ &\le \lambda_1(E_\tau)dq(h) + (1-\lambda_1(E_\tau))2\|h\| \\ &\le dq(h) + 2d\|h\|/(2\alpha) \le 2dq(h),\end{aligned}$$

$$|f(x+\tau e+h) - f(x+\tau e) - \langle \xi, h\rangle| \le 2dq(h). \tag{10}$$

Finally, note that $x+\tau e$, $x+\tau e+h \in G$. So we can estimate

$$\begin{aligned}f(x+\tau e) &\ge f(x) + \tau\langle \xi, e\rangle - \beta\tau \\ &\ge f(x) + \tau(s-\beta) - \beta\tau \ge f(x) + \tau s - 2\beta t\end{aligned} \tag{11}$$

by (5) and (4). Also, there are $\rho_n \uparrow 1$ such that

$$\begin{aligned}f(x+\tau e+\rho_n h) = \phi(\tau, \rho_n) &= \phi(0, \rho_n) + \int_0^\tau \phi_1'(\theta, \rho_n)d\theta \\ &= f(x+\rho_n h) + \int_0^\tau V f(x+\theta e+\rho_n h)(e)d\theta \\ &\le f(x+\rho_n h) + \tau s,\end{aligned}$$

hence

$$f(x+\tau e+h) \le f(x+h) + \tau s.$$

Now by combining the last inequality with (11), we obtain

$$f(x+h) - f(x) > f(x+\tau e+h) - f(x+\tau e) - 2\beta t.$$

Thus, putting this inequality together with (10) and (6), we get

$$f(x+h) - f(x) - \langle \xi, h\rangle > -2dq(h) - dq(h) = -3dq(h).$$

So far we have been working in the parallelepiped with vertices $x$, $x+te$, $x+h$, and $x+te+h$. Performing an analogous reasoning in the parallelepiped with vertices $x$, $x-te$, $x+h$, and $x-te+h$ we can find a $\tilde\tau \in (0, t)$ such that

$$|f(x-\tilde\tau e+h) - f(x-\tilde\tau e) - \langle \xi, h\rangle| \le 2dq(h)$$

and

$$f(x+h) - f(x) < f(x-\tilde\tau e+h) - f(x-\tilde\tau e) + 2\beta t.$$

By combining the last two inequalities and (6) we get

$$f(x+h) - f(x) - \langle \xi, h\rangle < 3dq(h).$$

*Proof of Theorem.* Let $Z$ an Asplund space and $T : Z \to Y$ be a linear continuous mapping such that $T(Z)$ is dense in $Y$. We work in the subspace $X$

of $Y$. Let $\Omega$ be an open subset of $X$ and $f : \Omega \to \mathbb{R}$ be a locally Lipschitz function. According to [6, Chapter 1, §10] we know that a set $M \subset \Omega$ is residual in $\Omega$ if every point $x \in \Omega$ has a neighborhood $W$ such that $M \cap W$ is residual in $W$. It follows that we may assume without loss of generality, that $f$ is (globally) Lipschitz on the whole $\Omega$. Also, we may and do assume that the Lipschitz constant of $f$ is equal to 1.

For $n, p \in \mathbb{N}$ put

$$A_{n,p} = [nT(B_Z) + \tfrac{1}{p}B_Y] \cap X,$$

where $B_Z$, $B_Y$ denote the unit balls in $Z$, $Y$ respectively. Then we claim that *every subset in the unit ball* $B_{X^*}$ *of* $X^*$ *is* $3/p - A_{n,p}$*-dentable*$^*$. Assuming this fact, then the lemma says that the sets

$$M_{n,p} = \Big\{x \in \Omega : \exists \xi \in B_{X^*} \exists \Delta > 0 \ |f(x+h) - f(x) - \langle \xi, h\rangle| \le (9/p)q_{n,p}(h) \\ \text{whenever } h \in X \text{ and } q_{n,p}(h) < \Delta\Big\}$$

are dense in $\Omega$, where $q_{n,p}$ are the Minkowski's functionals of $A_{n,p}$. Put

$$\widetilde{M}_{n,p} = \{x \in \Omega : \exists \xi \in B_{X^*} \ \exists r \in (0, 1/p) \\ \sup\{|f(x+rh) - f(x) - \langle \xi, rh\rangle| : h \in A_{n,p}\} < 10r/p\}.$$

These sets are open because $f$ is Lipschitz. They are also dense in $\Omega$ since they contain $M_{n,p}$. Thus putting $M = \bigcap_{n,p=1}^{\infty} \widetilde{M}_{n,p}$, this set is residual in $\Omega$.

We show that $M$ is the set, which we are looking for. So fix some $x \in M$. For any sequence $\sigma = (n_1, n_2, \dots) \in \mathbb{N}^{\mathbb{N}}$ put $A_\sigma = \bigcap_{p=1}^{\infty} A_{n_p,p}$. Let us fix for a moment one such $\sigma$. Then for each $p \in \mathbb{N}$ we can find $r_p \in (0, 1/p)$ and $\xi_p \in B_{X^*}$ such that

$$\left|\frac{1}{r_p}[f(x + r_p h) - f(x)] - \langle \xi_p, h\rangle\right| < \frac{10}{p} \quad \text{whenever } h \in A_{n_p,p}.$$

As the sequence $\{\xi_p\}$ lies in $B_{X^*}$, it has a weak$^*$ cluster point $\xi_\sigma$, say. Fix for a moment any $h \in A_\sigma$. For each $i = 1, 2, \dots$ there is $p_i > i$ such that $|\langle \xi_{p_i} - \xi_\sigma, h\rangle| < 1/i$. Then

$$\begin{aligned}\left|\frac{1}{r_{p_i}}[f(x + r_{p_i} h) - f(x)] - \langle \xi_\sigma, h\rangle\right| &\\ \le \left|\frac{1}{r_{p_i}}[f(x + r_{p_i} h) - f(x)] - \langle \xi_{p_i}, h\rangle\right| &+ |\langle \xi_{p_i} - \xi_\sigma, h\rangle| \\ < 10/p_i + 1/i < 11/i \to 0 \quad &\text{as } i \to \infty.\end{aligned}$$

And, recalling that $0 < r_{p_i} < 1/p_i < 1/i$, we can conclude that

$$D^+ f(x, h) \ge \langle \xi_\sigma, h\rangle \ge D_+ f(x, h). \tag{12}$$

This, of course, holds for every $h \in A_\sigma$. Let $\psi_\sigma(x)$ denote the set of all the $\xi_\sigma$ fulfilling (12). We remark that if $\sigma, \tau \in \mathbb{N}^\mathbb{N}$, then there exists $\rho \in \mathbb{N}^\mathbb{N}$ such that $A_\sigma \cup A_\tau \subset A_\rho$; hence $\psi_\rho(x) \subset \psi_\sigma(x) \cap \psi_\tau(x)$. And as $\psi_\sigma(x)$ are weakly$^*$ closed and lie in $B_{X^*}$, we get that the intersection

$$\bigcap\{\psi_\sigma(x) : \sigma \in \mathbb{N}^\mathbb{N}\}$$

is nonempty. Take some $\xi$ in this set. Then

$$D^+ f(x, h) \geq \langle \xi, h \rangle \geq D_+ f(x, h) \quad \text{for all } h \in X.$$

since $\bigcup\{A_\sigma : \sigma \in \mathbb{N}^\mathbb{N}\} = X$.

It remains to prove the claim. So fix $n, p \in \mathbb{N}$ and let $E$ be a nonempty subset of $B_{X^*}$. Let $\widetilde{E}$ denote the weakly$^*$ closed convex hull of $E$. Following the proof of [7, Theorem 12], let $F$ be a weakly$^*$ closed convex subset of $B_{Y^*}$, minimal with respect to the inclusion, and such that

$$\widetilde{E} = \{\xi/X : \xi \in F\},$$

where $\xi/X$ means the restriction of $\xi$ to the space $X$. As $Z$ is Asplund, there is $z \in Z$ and $\beta > 0$ such that the slice $S(T^*(F), z, \beta)$ is nonempty and

$$\operatorname{diam}_{B_Z} S(T^*(F), z, \beta) < 1/(np).$$

Then $S(F, Tz, \beta) \neq \emptyset$ and we estimate

$$\begin{aligned} \operatorname{diam}_{nT(B_Z)} S(F, Tz, \beta) &= \sup\{\langle \xi_1 - \xi_2, v\rangle : \xi_i \in S(F, Tz, \beta), v \in nT(B_Z)\} \\ &= \sup\{\langle T^*\xi_1 - T^*\xi_2, u\rangle : T^*\xi_i \in S(T^*(F), z, \beta),\ u \in nB_Z\} \\ &= n \operatorname{diam}_{B_Z} S(T^*(F), z, \beta) < n/(np) = 1/p. \end{aligned}$$

Now the set

$$E_1 = \{\xi/X : \xi \in F \backslash S(F, Tz, \beta)\}$$

is weakly$^*$ closed, convex, and, by the minimality, $E_1 \neq \widetilde{E}$. So there are $0 \neq e \in X$ and $\alpha > 0$ such that $S(\widetilde{E}, e, \alpha) \neq \emptyset$ and

$$S(\widetilde{E}, e, \alpha) \cap E_1 = \emptyset.$$

Then $\xi \in S(F, Tz, \beta)$ whenever $\xi \in F$ and $\xi/X \in S(\widetilde{E}, e, \alpha)$. Hence

$$\begin{aligned} &\operatorname{diam}_{A_{n,p}} S(E, e, \alpha) \\ &\quad \leq \operatorname{diam}_{A_{n,p}} S(\widetilde{E}, e, \alpha) \\ &\quad = \sup\{\langle \eta_1 - \eta_2, v\rangle : \eta_i \in S(\widetilde{E}, e, \alpha),\ v \in [nT(B_Z) + \tfrac{1}{p}B_Y] \cap X\} \\ &\quad \leq \sup\{\langle \xi_1 - \xi_2, v\rangle : \xi_i \in S(F, Tz, \beta),\ v \in nT(B_Z) + \tfrac{1}{p}B_Y\} \\ &\quad \leq \sup\{\langle \xi_1 - \xi_2, v\rangle : \xi_i \in S(F, Tz, \beta), v \in nT(B_Z)\} + 2/p \\ &\quad = \operatorname{diam}_{nT(B_Z)} S(F, Tz, \beta) + 2/p < 3/p. \end{aligned}$$

Finally we recall that $\widetilde{E}$ is a weakly$^*$ closed convex hull of $E$, so $S(\widetilde{E}, e, \alpha) \neq \emptyset$ implies that $S(E, e, \alpha)$ is also nonempty. This finishes the proof of the claim.

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