PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 113, Number 3, November 1991

A NONUNIFORM VERSION OF THE THEOREM OF RADON-NIKODYM IN THE FINITELY-ADDITIVE CASE WITH APPLICATIONS TO EXTENSIONS OF FINITELY-ADDITIVE SET FUNCTIONS

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(Communicated by Lawrence F. Gray)

ABSTRACT. For μ, $\nu \in ba_+(\Omega, \mathfrak{A})$ it is shown that the existence of a net of nonnegative functions f_α, that are primitive relative to $\mathfrak{A}$ and satisfy $\lim_\alpha \int_A f_\alpha \, d\mu = \nu(A)$, $A \in \mathfrak{A}$, is equivalent to the condition $\nu \lesssim \mu$, i.e. $\mu(A) = 0$ for some $A \in \mathfrak{A}$ implies $\nu(A) = 0$. Furthermore, as an application it is proved that for μ, $\nu \in ba_+(\Omega, \mathfrak{A})$ satisfying $\nu \lesssim \mu$ and any extension $\mu' \in ba_+(\Omega, \mathfrak{A}')$ of μ, where $\mathfrak{A}'$ denotes some algebra of subsets of Ω containing $\mathfrak{A}$, there exists some extension $\nu' \in ba_+(\Omega, \mathfrak{A}')$ of ν such that $\nu' \lesssim \mu'$ is valid.

Let $ba_+(\Omega, \mathfrak{A})$ denote the subset of the set $ba(\Omega, \mathfrak{A})$ of all bounded and finitely-additive set functions on the algebra $\mathfrak{A}$ of subsets of the set Ω that are nonnegative. Furthermore, $B(\Omega, \mathfrak{A})$ stands for the family of all real-valued functions on Ω that are uniform limits of functions of the type $\sum_{i=1}^n \alpha_i I_{A_i}$, $\alpha_i \in \mathbb{R}$, $A_1 \in \mathfrak{A}$, $i = 1, \ldots, n$, $n \in \mathbb{N}$ ("primitive function relative to $\mathfrak{A}$"). Finally, ν is called weakly absolutely continuous with respect to μ (symbol: $\nu \lesssim \mu$), if $\mu(A) = 0$ for some $A \in \mathfrak{A}$ implies $\nu(A) = 0$, and ν is called absolutely continuous with respect to μ (symbol: $\nu \ll \mu$), if for any $\varepsilon > 0$ there exists some $\delta > 0$ such $\nu(A) < \varepsilon$ whenever $\mu(A) < \delta$ for $A \in \mathfrak{A}$, where μ, ν belong to $ba_+(\Omega, \mathfrak{A})$ (cf. [1, p. 159]). In the sequel the following equivalent condition for the relation $\nu \lesssim \mu$, μ, $\nu \in ba_+(\Omega, \mathfrak{A})$, is used: $\int_A f \, d\mu \geq 0$ for all $A \in \mathfrak{A}$ for some $f \in B(\Omega, \mathfrak{A})$ implies $\int_A f \, d\nu \geq 0$ for all $A \in \mathfrak{A}$. This last condition is equivalent to the relation $\nu \lesssim \mu$, since one can replace $f \in B(\Omega, \mathfrak{A})$ by some function $f: \Omega \to \mathbb{R}$, which is primitive relative to $\mathfrak{A}$.

According to a well-known uniform version of the Radon-Nikodym theorem in the finitely-additive case due to Bochner, there exists a net f_α of nonnegative

Received by the editors February 8, 1990 and, in revised forms, May 20, 1990 and August 15, 1990.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 28A12, 28A20.

Key words and phrases. Nonuniform version of the Radon-Nikodym theorem, finitely-additive set functions.

functions that are primitive relative to $\mathfrak{A}$ and satisfy $\lim_\alpha \int_A f_\alpha \, d\mu = \nu(A)$ uniformly for $A \in \mathfrak{A}$ if and only if $\nu \ll \mu$ is valid (cf. [1, Theorem 6.3.4, p. 175] or [2, IV.9.14, p. 315]). Clearly, one might replace the net f_α by a sequence, whereas the notion of a net is essential for the following nonuniform version of the Radon-Nikodym theorem in the finitely-additive case:

Theorem. *For μ, $\nu \in ba(\Omega, \mathfrak{A})$ there exists a net of nonnegative functions f_α that are primitive relative to $\mathfrak{A}$ and satisfy $\lim_\alpha \int_A f_\alpha \, d\mu = \nu(A)$ for $A \in \mathfrak{A}$, if and only if $\nu \lesssim \mu$ is valid.*

Proof. Clearly for a net f_α of nonnegative functions that are primitive relative to $\mathfrak{A}$, $\lim_\alpha \int_A f_\alpha \, d\mu = \nu(A)$, $A \in \mathfrak{A}$, implies $\nu \lesssim \mu$. For the converse direction let C denote the convex subset of $ba_+(\Omega, \mathfrak{A})$ consisting of all $\mu_f \in ba_+(\Omega, \mathfrak{A})$ defined by $\mu_f(A) = \int_A f \, d\mu$, $A \in \mathfrak{A}$, where f belongs to the system $\mathfrak{F}$ of all nonnegative functions that are primitive with respect to $\mathfrak{A}$. Now the assumption that ν does not belong to the closure of C with respect to the $B(\Omega, \mathfrak{A})$-topology of $ba(\Omega, \mathfrak{A})$ (cf. [1, Theorem 4.7.4] or [2, IV.5.1, p. 258]), implies the existence of some $g \in B(\Omega, \mathfrak{A})$ with the property $\int fg \, d\mu \le c < c+\varepsilon \le \int g \, d\nu$ for all $f \in \mathfrak{F}$ and some $c \in \mathbb{R}$ and $\varepsilon > 0$, if one applies a well-known separation theorem for convex subsets (cf. [2, V.2.10, p. 417]) together with [2, V.3.12, p. 422]. In particular, $\int fg \, d\mu \le 0$ for $f \in \mathfrak{F}$ is valid, since $\int fg \, d\mu > 0$ for some $f \in \mathfrak{F}$ implies $\sup\{\int fg \, d\mu : f \in \mathfrak{F}\} = \infty$. Furthermore, $\int_A g \, d\mu \le 0$, $A \in \mathfrak{A}$, implies $\int_A g \, d\nu \le 0$, $A \in \mathfrak{A}$, because of $\nu \lesssim \mu$. This is a contradiction to $\int g \, d\nu \ge c+\varepsilon \ge \varepsilon$, since $c \ge 0$ is implied by $\int fg \, d\mu \le c < c+\varepsilon \le \int g \, d\nu$, $f \in \mathfrak{F}$, if one chooses $f = 0$. □

Remark. Under the additional assumption that $\mathfrak{A}$ is a σ-algebra of subsets of Ω (or more generally an algebra of subsets of Ω with the Seever property, cf. [1, p. 210]), the net of nonnegative functions f_α of the theorem can be replaced by a sequence if and only if the classical uniform version of the theorem of Radon-Nikodym (cf. [1, Theorem 6.3.4, p. 175] or [2, IV.9.14, p. 315]) is valid, i.e. $\nu \ll \mu$ holds true (cf. [1, Theorem 8.7.5, p. 225] and [2, V.3.14, p. 422]). The conclusion $\nu \ll \mu$ is no longer true if $\mathfrak{A}$ is only an algebra, as the special case shows, where $\mathfrak{A}$ is countable. In this situation $B(\Omega, \mathfrak{A})$ is separable, which implies that the underlying net of functions of the theorem may be replaced by a sequence (cf. [2, V.5.1, p. 426]).

A simple application of the theorem concerns the existence of extensions of μ, $\nu \in ba_+(\Omega, \mathfrak{A})$ according to the following:

Corollary. *For μ, $\nu \in ba_+(\Omega, \mathfrak{A})$ satisfying $\nu \lesssim \mu$ and any extension $\mu' \in ba_+(\Omega, \mathfrak{A}')$ of μ, where $\mathfrak{A}'$ is some algebra of subsets of Ω containing $\mathfrak{A}$, there exists an extension $\nu' \in ba_+(\Omega, \mathfrak{A}')$ of ν such that $\nu' \lesssim \mu'$ is valid.*

Proof. Let f_α denote a net of nonnegative functions that are primitive relative to $\mathfrak{A}$ and satisfy $\lim_\alpha \int_A f_\alpha \, d\mu = \nu(A)$, $A \in \mathfrak{A}$. Now the compactness of the unit ball of $ba(\Omega, \mathfrak{A}')$ with respect to the $B(\Omega, \mathfrak{A}')$-topology of $ba(\Omega, \mathfrak{A}')$ implies the existence of some subnet f_β of the net f_α such that $\lim_\beta \int_{A'} f_\beta \, d\mu'$

exists for all $A' \in \mathfrak{A}'$ (cf. [2, V.4.2, p. 424] together with [2, IV.5.1, p. 258]), which defines some extension $\nu' \in ba_+(\Omega, \mathfrak{A}')$ of ν satisfying $\nu' \lesssim \mu'$. □

It might be interesting to find conditions that imply that the f_α of the underlying net of nonnegative functions of the theorem can be chosen to be already primitive relative to some given subalgebra $\overline{\mathfrak{A}}$ of $\mathfrak{A}$. The first example provides a sufficient condition of this type in terms of extensions of the restriction $\nu|\overline{\mathfrak{A}}$ of $\nu \in ba_+(\Omega, \mathfrak{A})$ to $\overline{\mathfrak{A}}$.

Example 1. Let μ, $\nu \in ba_+(\Omega, \mathfrak{A})$ satisfy $\nu \lesssim \mu$. Then there exists a net f_α of nonnegative functions that are primitive relative to $\overline{\mathfrak{A}}$, where $\overline{\mathfrak{A}}$ denotes some subalgebra of $\mathfrak{A}$, such that $\lim_\alpha \int_A f_\alpha \, d\mu = \nu(A)$, $A \in \mathfrak{A}$, is valid, if ν is the uniquely determined extension of $\nu|\overline{\mathfrak{A}}$ to $\mathfrak{A}$ among all $\lambda \in ba_+(\Omega, \mathfrak{A})$ satisfying $\lambda \lesssim \mu$ and $\lambda|\overline{\mathfrak{A}} = \nu|\overline{\mathfrak{A}}$. This follows similar to the proof of the corollary by applying [2, V.4.2, p. 424, IV.5.1, p. 258] to some net f_α of nonnegative functions that are primitive relative to $\overline{\mathfrak{A}}$, and satisfy $\lim_\alpha \int_A f_\alpha \, d\mu = \nu(A)$ for all $A \in \overline{\mathfrak{A}}$. Thus one gets a subnet f_β of f_α such that $\lim_\beta \int_A f_\beta \, d\mu$ exists for all $A \in \mathfrak{A}$, i.e. $\lambda(A) = \lim_\beta \int_A f_\beta \, d\mu$, $A \in \mathfrak{A}$, defines some $\lambda \in ba_+(\Omega, \mathfrak{A})$ satisfying $\lambda \lesssim \mu$ and $\lambda|\overline{\mathfrak{A}} = \nu|\overline{\mathfrak{A}}$. Therefore, $\lambda = \nu$ is valid. The method of proof also yields the following characterization of the property of $\nu \in ba_+(\Omega, \mathfrak{A})$ to be uniquely determined among all $\lambda \in ba_+(\Omega, \mathfrak{A})$ satisfying $\lambda \lesssim \mu$ and $\lambda|\overline{\mathfrak{A}} = \nu|\overline{\mathfrak{A}}$: For any net f_α of nonnegative functions that are primitive relative to $\mathfrak{A}$ and satisfy $\lim_\alpha \int_A f_\alpha \, d\mu = \nu(A)$ for all $A \in \overline{\mathfrak{A}}$ the limit $\lim_\alpha \int_A f_\alpha \, d\mu$ exists for any $A \in \mathfrak{A}$ and is equal to $\nu(A)$, $A \in \mathfrak{A}$.

The second example concerns the special subalgebra $\overline{\mathfrak{A}}$ of invariant sets of $\mathfrak{A}$ in connection with the problem of finding conditions, which imply that the f_α of the underlying net of nonnegative functions of the theorem can be chosen to be already primitive relative to $\overline{\mathfrak{A}}$.

Example 2. Let Γ denote a finite group of transformations $\gamma : \Omega \to \Omega$ satisfying $\gamma^{-1}(A) \in \mathfrak{A}$, $A \in \mathfrak{A}$, such that μ, $\nu \in ba_+(\Omega, \mathfrak{A})$ are Γ-invariant, where $\lambda \in ba_+(\Omega, \mathfrak{A})$ is said to be Γ-invariant if and only if $\lambda(\gamma^{-1}(A)) = \lambda(A)$, $A \in \mathfrak{A}$, holds true. Under this general assumption there exists a net f_α of nonnegative functions that are primitive relative to $\overline{\mathfrak{A}}$, where $\overline{\mathfrak{A}}$ denotes the subalgebra $\{A \in \mathfrak{A} : \gamma(A) = A, \ \gamma \in \Gamma\}$ of Γ-invariant sets belonging to $\mathfrak{A}$, and satisfy $\lim_\alpha \int_A f_\alpha \, d\mu = \nu(A)$, $A \in \mathfrak{A}$ if and only if $\nu \lesssim \mu$ is valid. This might be seen by starting from the inequality $\int fg \, d\mu \le c < c + \varepsilon \le \int g \, d\nu$ for all $f \in \mathfrak{F}$ of the first part of the proof of the theorem, where $\mathfrak{F}$ now stands for the set of all nonnegative functions that are primitive relative to $\overline{\mathfrak{A}}$. The general assumption implies that $g \in B(\Omega, \mathfrak{A})$ might be replaced by the function $h \in B(\Omega, \overline{\mathfrak{A}})$, which is defined by $h = \sum_{\gamma \in \Gamma} h \circ \gamma / |\Gamma|$ ($|\Gamma|$ cardinality of Γ). The remaining part of the proof is similar to the last part of the proof for the theorem. Example 2 admits as an application a characterization of the property of a Γ-invariant $\mu \in ba_+(\Omega, \mathfrak{A})$, $\mu(\Omega) = 1$, that Γ's action is ergodic relative to μ, i.e. $\mu|\overline{\mathfrak{A}}$ is $\{0, 1\}$-valued. This property of μ is equivalent to the condition that there does not exist some Γ-invariant $\nu \in ba_+(\Omega, \mathfrak{A})$ satisfying $\nu \neq \mu$, $\nu(\Omega) = 1$, and $\nu \lesssim \mu$. Clearly, if $\mu|\overline{\mathfrak{A}}$ is not $\{0, 1\}$-valued, then ν defined by $\nu(A) = \mu(A \cap A_0)/\mu(A_0)$, $A \in \mathfrak{A}$, where for $A_0 \in \overline{\mathfrak{A}}$ the inequality $0 < \mu(A_0) < 1$ is valid, is Γ-invariant and satisfies $\nu \neq \mu$, $\nu \lesssim \mu$. Conversely, $\lim_\alpha \int_A f_\alpha \, d\mu = \nu(A)$, $A \in \mathfrak{A}$, where f_α is a net of nonnegative functions that are primitive relative to $\overline{\mathfrak{A}}$, implies $\mu(\{f_\alpha = c_\alpha\}) = 1$ for some $c_\alpha \geq 0$, from which $\mu(A) = \nu(A)$, $A \in \mathfrak{A}$, follows.

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