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FIXED POINT ITERATION FOR LOCAL STRICTLY PSEUDO-CONTRACTIVE MAPPING

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ABSTRACT. A fixed point of the local strictly pseudo-contractive mapping is obtained as the limit of an iteratively constructed sequence with an error estimation in uniformly smooth Banach spaces.

This paper has been motivated by a paper by C. E. Chidume [2], in which an iterative approximation of the fixed point of a Lipschitzian strictly pseudo-contractive mapping in L_p $(2 \le p < \infty)$ space is developed. The object of the present note is to consider the local version of the problem and to extend Chidume's results to general uniformly smooth Banach spaces. Furthermore, we replace strictly pseudo-contractive mapping by local strictly pseudo-contractive mapping and the Lipschitzian condition by a weaker assumption.

Let X be a Banach space, a mapping T is said to be local strictly pseudo-contractive if for any $x \in D(T)$ there exists a number $t_x > 1$ such that the inequality

$$\|x - y\| \le \|(1 + r)(x - y) - rt_x(Tx - Ty)\|$$

holds for all $y \in D(T)$ and $r > 0$. We denote by J the normalized duality mapping from X to 2^{X^*} given by

$$Jx = \{f^* \in X^* : \|f^*\|^2 = \|x\|^2 = \operatorname{Re}\langle x, f^*\rangle\}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. If X is uniformly smooth, then J is single-valued and uniformly continuous on bounded set. A mapping T is said to be local strongly accretive if given $x \in D(T)$ there exists a positive number k_x such that for each $y \in D(T)$ there is $j \in J(x - y)$ such that

$$\langle Tx - Ty, j\rangle \ge k_x\|x - y\|^2.$$

In [3] T. Kato discovered the relationship between pseudo-contractive mappings

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and accretive mappings, proving

Lemma 1 [3]. *Let* $x, y \in X$. *Then* $\|x\| \le \|x + \alpha y\|$ *for every* $\alpha > 0$ *if and only if there is* $f \in Jx$ *such that* $\operatorname{Re}\langle y, f\rangle \ge 0$.

Applying Lemma 1, we get

Lemma 2. *Let* X *be a Banach space,* K *a subset of* X, *and* $U: K \mapsto X$. *Then* U *is a local strictly pseudo-contractive mapping if and only if* $T = I - U$ *is a local strongly accretive mapping.*

Proof. If U is a local strictly pseudo-contractive mapping, then given any $x \in K$ there exists a number $t_x > 1$ such that for any $y \in K$, $r > 0$ we have

$$\begin{aligned}\|x - y\| &\le \|(1+r)(x-y) - rt_x(Ux - Uy)\| \\ &= \|(1+r)(x-y) - r(t_xUx - t_xUy)\| \\ &= \|x - y + r(x - t_xUx - y + t_xUy)\|.\end{aligned}$$

By Lemma 1, there exists $j \in J(x-y)$ such that

$$\operatorname{Re}\langle x - t_xUx - y + t_xUy, j\rangle \ge 0.$$

Observe that operator $I - t_xU = I - t_x(I - T) = t_xT - (t_x - 1)I$ we obtain

$$t_x \operatorname{Re}\langle Tx - Ty, j\rangle - (t_x - 1)\operatorname{Re}\langle x - y, j\rangle \ge 0,$$

which produces

$$\operatorname{Re}\langle Tx - Ty, j\rangle \ge (1 - 1/t_x)\|x - y\|^2$$

for all $y \in K$, i.e., T is a local strongly accretive mapping. Suppose, conversely, that T is local strongly accretive. Then for any given $x \in K$ there exists a number $k_x > 0$ and for $y \in K$ there exists $j \in J(x-y)$ such that

$$\operatorname{Re}\langle Tx - Ty, j\rangle \ge k_x\|x - y\|^2.$$

Without loss of generality, we can assume $1 > k_x > 0$, and let $t_x = 1/(1 - k_x)$. Then the above inequality becomes

$$\operatorname{Re}\langle Tx - Ty, j\rangle \ge (1 - 1/t_x)\|x - y\|^2.$$

By the same proof as above but in the reverse order, we get that U is a local strictly pseudo-contractive mapping.

Let X be an arbitrary Banach space with $\dim X \ge 2$, the modulus of convexity $\delta_X(\varepsilon)$, $0 < \varepsilon \le 2$, of X is defined by

$$\delta_X(\varepsilon) = \inf\{1 - \|x + y\|/2 : x, y \in X,\ \|x\| = \|y\| = 1,\ \|x - y\| = \varepsilon\}.$$

X is said to be uniformly convex if $\delta_X(\varepsilon) > 0$ for every $\varepsilon > 0$ and uniformly smooth if the dual space X^* is uniformly convex. The estimation of the modulus of convexity for the spaces L^p, e^p, W_m^p, $1 < p < \infty$ are [1]:

$$\begin{aligned}\delta_X(\varepsilon) &\ge (p - 1/16)\varepsilon^2, \qquad 1 < p \le 2, \\ \delta_X(\varepsilon) &\ge p^{-1}(\varepsilon/2)^p, \qquad p \ge 2.\end{aligned}$$

We define for positive t

$$\beta(t) = \sup\{(\|x + ty\|^2 - \|x\|^2)/t - 2\operatorname{Re}\langle y, J(x)\rangle : \|x\| \le 1, \ \|y\| \le 1\}$$

Clearly $\beta : (0, \infty) \to [0, \infty)$ is nondecreasing, continuous and $\beta(ct) \le c\beta(t)$ for $c \ge 1$. Also we have

Lemma 3 [4]. *If X is a uniformly smooth Banach space and $\beta(t)$ is defined as above, then $\lim_{t\to 0+} \beta(t) = 0$ and*

$$\|x + y\|^2 \le \|x\|^2 + 2\operatorname{Re}\langle y, J(x)\rangle + \max\{\|x\|, 1\}\|y\|\beta(\|y\|)$$

for all $x, y \in X$.

We also need the following:

Lemma 4. *Let β_n be a nonnegative sequence satisfying*

$$\beta_{n+1} \le (1 - \delta_n)\beta_n + \sigma_n$$

with $\delta_n \in [0, 1]$, $\sum_{i=1}^{\infty} \delta_i = \infty$, and $\sigma_n = o(\delta_n)$. Then $\lim_{n\to\infty} \beta_n = 0$.

Proof. Since $\sigma_n = o(\delta_n)$, let $\sigma_n = \varepsilon_n \cdot \delta_n$, and $\varepsilon_n \to 0$. By a straightforward induction, one obtains

$$(*) \qquad 0 \le \beta_{n+1} \le \prod_{j=k}^{n}(1 - \delta_j)\beta_k + \sum_{j=k}^{n}\left[\delta_j \prod_{i=j+1}^{n}(1 - \delta_i)\right]\varepsilon_j .$$

We have

$$\prod_{j=k}^{n}(1 - \delta_j) \le e^{-\sum_{j=k}^{n}\delta_j} \to 0$$

and

$$\sum_{j=k}^{n}\delta_j \prod_{i=j+1}^{n}(1 - \delta_i) \le 1 \qquad \forall n, k .$$

Given $\varepsilon > 0$, pick k such that $\varepsilon_j \le \varepsilon$ for all $j \ge k$, from $(*)$ we have

$$0 \le \liminf \beta_n \le \limsup \beta_n \le \varepsilon .$$

Letting $\varepsilon \to 0$, we obtain $\lim_{n\to\infty} \beta_n = 0$.

For the rest of this note, $F(U)$ denotes the set of fixed points of the mapping U.

Theorem 1. *Let K be a subset of a uniformly smooth Banach space and $U : K \mapsto X$ be a local strictly pseudo-contractive mapping. If $F(U) \ne \emptyset$ and the range of U is bounded, then $\{x_n\} \subset K$ generated by $x_1 \in K$,*

$$x_{n+1} = (1 - C_n)x_n + C_n U x_n$$

with $\{C_n\} \subset (0, 1]$, satisfying

$$\sum_{n=1}^{\infty} C_n = \infty, \qquad C_n \to 0$$

strongly converges to $p \in F(U)$ and $F(U)$ is a single set.

Proof. Let p be a fixed point of U. Since U is local strictly pseudo contractive mapping, then $T = I - U$ is local strongly accretive. Thus, there exists a positive number k_p such that for each $x_n \in K$

$$\operatorname{Re}\langle x_n - Ux_n - p + Up, J(x_n - p)\rangle \geq k_p \|x_n - p\|^2.$$

Now set

$$\beta_n = \|x_n - p\|^2$$

and

$$d = \sup\{\|Ux - p\| : x \in K\}.$$

Because $C_n \to 0$, it is easy to show there exists an integer $N \geq 1$ such that when $n \geq N$

$$[1 - k_p C_n]^2 + d^2 C_n \beta(C_n) \leq 1.$$

Let $B = \max\{\beta_i : 1 \leq i \leq N, 1\}$. First we want to show $\beta_n \leq B^2$, and

$$(**) \qquad \beta_{n+1} \leq [1 - k_p C_n]^2 \beta_n + B^2 d^2 C_n \beta(C_n).$$

From the definition of $\{x_n\}$ and Lemma 3, we have

$$\begin{aligned}
\beta_{n+1} &= \|x_{n+1} - p\|^2 = \|(1 - C_n)(x_n - p) + C_n(Ux_n - p)\|^2 \\
&\leq (1 - C_n)^2 \|x_n - p\|^2 + 2C_n(1 - C_n)\operatorname{Re}\langle Ux_n - p, J(x_n - p)\rangle \\
&\quad + \max\{\|x_n - p\|, 1\} C_n \|Ux_n - p\| \beta(C_n \|Ux_n - p\|) \\
&\leq (1 - C_n)^2 \beta_n + \max\{\|x_n - p\|, 1\} d^2 C_n \beta(C_n) \\
&\quad + 2C_n(1 - C_n)\|x_n - p\|^2 - 2C_n(1 - C_n) \\
&\quad \times \operatorname{Re}\langle x_n - Ux_n - p + Up, J(x_n - p)\rangle \\
&\leq [(1 - C_n)^2 + 2(1 - k_p)C_n(1 - C_n)]\beta_n + \max\{\|x_n - p\|, 1\} d^2 C_n \beta(C_n) \\
&\leq [1 - k_p C_n]^2 \beta_n + \max\{\beta_n, 1\} d^2 C_n \beta(C_n).
\end{aligned}$$

If $n \leq N$, according the definition of number B, so $\beta_n \leq B^2$. For $n \geq N$ apply induction: assume $\beta_n \leq B^2$ then

$$\begin{aligned}
\beta_{n+1} &\leq [1 - k_p C_n]^2 \beta_n + B^2 d^2 C_n \beta(C_n) \\
&\leq \{[1 - k_p C_n]^2 + d^2 C_n \beta(C_n)\} B^2 \leq B^2,
\end{aligned}$$

so we obtain $\beta_n \leq B^2$ for all n and from above inequality we get $(**)$. Now applying Lemma 4 to inequality $(**)$ we get that $\{x_n\}$ strongly converges to the fixed point p. Now suppose there is a $p^* \in F(U)$ with $p^* \neq p$. Repeating the argument of the theorem relative to p^*, one sees that $\{x_n\}$ converges to both p^* and p, showing that $F(U) = \{p\}$.

For the special Banach spaces L^p, e^p, W_m^p, $1 < p < \infty$, we have the estimate [5]:

$$\beta(t) \leq Mt^{s-1},$$

$s = 2$ if $2 \leq p < \infty$, $s = p$ if $1 < p < 2$, and M is some constant. Then we may able to obtain a convergence rate in the setting of Theorem 1.

Theorem 2. *Let $X = L^p$, e^p, W_m^p, $1 < p < \infty$, and U, $\{x_n\}$, C_n be as in Theorem 1. Then we can find $\{C_n^*\}$ such that for sequence $\{x_n^*\}$, we have the estimate*

$$\|x_n^* - p\| \leq O(1/n^{(s-1)/2}),$$

where $s = 2$ if $2 \leq p < \infty$, $s = p$ if $1 < p < 2$.

Proof. Let

$$C_n^* = \frac{k_p^{1/(s-1)}}{1 + nk_p^{s/s(s-1)}},$$

$$x_{n+1}^* = (1 - C_n^*)x_n^* + C_n^* U x_n^*,$$

$$\beta_n^* = \|x_n^* - p\|^2,$$

and

$$\alpha_{n+1}^* = [1 - k_p C_n^*]^s \alpha_n^* + C_n^{*s}, \qquad \alpha_1^* = (MB^2 d^2)^{-1} \beta_1^*.$$

From $(**)$ and the estimation of $\beta(t)$ in X spaces, we have $\beta_n^* \leq \alpha_n^*$, $n = 1, 2, \ldots$. Solving the above equations we get

$$\alpha_n^* = \alpha_1^* / [1 + (n-1)\alpha_1^{*1/(s-1)} (k_p)^{s/(s-1)}]^{s-1}.$$

The proof is complete.

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