PRODUCT SHIFTS ON $B(H)$

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Abstract. A shift on $B(H)$ is a *-endomorphism $\alpha$ for which $\bigcap_n \alpha^n(B(H)) = CP$ for some projection $P$. The paper discusses some aspects of the classification of shifts on $B(H)$ up to conjugacy by *-automorphisms, with a focus on the shifts arising from an infinite tensor product decomposition of $H$.

1. Introduction

Let $H$ be a separable Hilbert space and let $B(H)$ be the algebra of all bounded linear operators on $H$. Slightly extending the definition given in [7] to include the nonunital case, a *-endomorphism $\alpha$ of $B(H)$ will be called a shift if $\bigcap_n \alpha^n(B(H)) = CP$ for some (possibly zero) projection $P$ on $B(H)$. The present paper discusses some aspects of the classification of shifts on $B(H)$ up to conjugacy by *-automorphisms, with a focus on the shifts arising from an infinite tensor product decomposition of $H$.

2. Spatial descriptions

It was observed in [2, Proposition 2.1] that every nonzero normal *-endomorphism $\alpha$ of $B(H)$ is implemented by a sequence of isometries. An alternative proof of this result can be obtained by imitating the usual proof that *-automorphisms of $B(H)$ are inner, as described, for example, in [5, Lemma 7.5.3], and this alternative proof shows that the normality assumption can be omitted (and hence every *-endomorphism of $B(H)$ is normal). For later reference we now state this minor extension of [2, Proposition 2.1].

Proposition 2.1. Let $\alpha$ be a nonzero *-endomorphism of $B(H)$. Then there is a (finite or infinite) sequence of isometries $V_1, V_2, \ldots$ in $B(H)$ having mutually orthogonal ranges such that

$$\alpha(A) = \sum_n V_n A V_n^*$$

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for each \( A \in B(H) \). The linear space of operators
\[
E_\alpha = \{ T \in B(H) : \alpha(A)T = TA \text{ for all } A \in B(H) \}
\]
is a Hilbert space relative to the inner product defined by
\[
T^*S = \langle S, T \rangle
\]
and \( \{ V_1, V_2, \ldots \} \) is an orthonormal basis for \( E_\alpha \).

The following simple result relates the conjugacy of \( \alpha \) and \( \beta \) to the Hilbert spaces \( E_\alpha \) and \( E_\beta \).

**Proposition 2.2.** Let \( \alpha, \beta \) be \(*\)-endomorphisms of \( B(H) \) and let \( U \in B(H) \) be unitary. Then \( \alpha = (\text{Ad } U)\beta(\text{Ad } U^*) \) if and only if \( E_\alpha = UE_\beta U^* \).

**Proof.** If \( \alpha = (\text{Ad } U)\beta(\text{Ad } U^*) \) then an easy calculation yields \( E_\alpha = UE_\beta U^* \). This same result shows that if \( E_\alpha = UE_\beta U^* \) then \( E_\alpha = E_\gamma \) where \( \gamma = (\text{Ad } U)\beta(\text{Ad } U^*) \); hence there exist orthonormal bases \( \{ V_1, V_2, \ldots \} \) and \( \{ W_1, W_2, \ldots \} \) of \( E_\alpha \) with \( \alpha(A) = \sum V_nAV_n^* \) and \( \gamma(A) = \sum W_nAW_n^* \) for each \( A \in B(H) \). A simple calculation using the unitary transition matrix between the two orthonormal bases then yields the conclusion \( \alpha = \gamma \). \( \square \)

A natural question arising from Propositions 2.1 and 2.2 is the extent to which the specification of the isometries \( V_i \) up to unitary equivalence determines the corresponding \(*\)-endomorphism up to conjugacy. The classification of an isometry \( V \) up to unitary equivalence is given by the Wold decomposition, described, for example, in [4, Problem 118], which decomposes \( V \) as a direct sum of copies of the unilateral shift and a unitary mapping on a subspace; when the unitary summand is absent the isometry is said to be pure. It will now be shown that if \( \alpha \) is a shift, then at most one of any corresponding family of isometries is not pure; this result extends [1, Theorem 3] which is the case of two unitarily equivalent isometries. The proof is based on the following lemma in which Cuntz's notation \( V_{\mu} V_{\nu}^* \) is used to denote \( V_{\mu(1)} \cdots V_{\mu(r)} V_{\nu(s)}^* \cdots V_{\nu(1)}^* \), where \( \mu = (\mu(1), \ldots, \mu(r)) \) has \( r = |\mu| \) components and \( \nu = (\nu(1), \ldots, \nu(s)) \) has \( s = |\nu| \) components.

**Lemma 2.3.** Let \( V_1, V_2, \ldots \) be isometries from a Hilbert space \( H \) onto a family of mutually orthogonal subspaces; \( \alpha \) be the \(*\)-endomorphism defined by \( \alpha(A) = \sum_n V_nAV_n^* \); and \( h \in \bigcap_i V_i^*H \).

(i) If \( P_h \) is the orthogonal projection from \( H \) onto the closed linear span \( S \) of the vectors \( V_{\mu} V_{\nu}^* \) with \( |\mu| \geq 0 \) and \( |\nu| \geq 0 \), then \( \alpha(P_h) = P_h \) and hence \( P_h \in \bigcap_i \alpha'(B(H)) \).

(ii) If \( Q_h \) is the orthogonal projection from \( H \) onto the closed linear span \( T \) of the vectors \( V_{\mu} V_{\nu}^* \) with \( |\mu| = |\nu| \geq 0 \), then \( Q_h \in \bigcap_i \alpha'(B(H)) \).

**Proof.** (i) Let \( |\mu| = r, |\nu| = s \), and \( h = V_i^{s+1}k \), where \( k \in H \). Then \( \alpha(1) V_{\mu} V_{\nu}^* h = \sum V_n V_{\mu} V_{\nu}^* V_i^{s+1}k = V_{\mu} V_{\nu}^* h \) (even if \( r = 0 \)) and hence \( \alpha(1) \geq P_h \). Since \( S \) is invariant under both \( V_n \) and \( V_{\nu}^* \), it follows that \( P_h V_n = V_n P_h \)
for each $n$ and hence $\alpha(P_h) = \sum V_n P_h V_n^* = P_h \sum V_n V_n^* = P_h \alpha(1) = P_h$.

(ii) Since $T$ is invariant under each $V_\gamma V_\delta^*$ with $|\gamma| = |\delta|$, it follows that $Q_h V_\gamma V_\delta^* = V_\gamma V_\delta^* Q_h$. Thus

$$
\alpha'(V_1 V_2^* Q_h V_1^*) = \sum_{|\gamma|=r} V_\gamma V_1 V_1^* V_1^* \alpha' V_\gamma V_1^* V_1^* \gamma = \sum_{|\gamma|=r} V_\gamma V_1 V_1^* V_\gamma^* Q_h = \sum_{|\gamma|=r} V_\gamma V_\gamma^* Q_h = \alpha'(1) Q_h.
$$

However, using part (i), $Q_h \leq P_h = \alpha'(P_h) \leq \alpha'(1)$ so $Q_h = \alpha'(V_1 V_2^* Q_h V_1^*)$ and hence $Q_h \in \cap_r \alpha'(B(H))$. □

Proposition 2.4. Let $V_1, V_2, \ldots$ be a sequence of isometries from a Hilbert space $H$ onto a family of orthogonal subspaces and let $\alpha$ be the $*$-endomorphism of $B(H)$ defined by $\alpha(A) = \sum_n V_n^* A V_n^*$. If $\alpha$ is a shift, then at most one of the isometries $V_n$ is not pure and if $V_n$ is not pure then the subspace $\cap_r V_n^* H$ associated with the unitary summand of $V_n$ is one-dimensional.

Proof. Suppose, to obtain a contradiction, that there exist nonzero vectors $h \in \cap_i V_i^* H$ and $k \in \cap_r V_r^* H$ with $\langle h, k \rangle = 0$. Then, by part (ii) of the lemma, there exist associated nonzero projections $Q_h, Q_k$ in $\cap_r \alpha'(B(H))$. Let $|\mu| = |\nu|$ and $|\delta| = |\gamma|$. The proof will be completed if it is shown that $\langle V_\mu V_\gamma V_\nu^* h, V_\gamma V_\delta^* k \rangle = 0$ for then $Q_h Q_k = 0$ and, therefore, $\alpha$ is not a shift. To this end let $V_\delta V_\nu V_\gamma V_\gamma^* V_\gamma^* V_\nu^* V_\mu^* = V_\rho V_\sigma^*; |\rho| = |\sigma| = r; h = V_i^r h'$; and $k = V_j^{r+1} k'$. Then $\langle V_\mu V_\gamma V_\nu^* h, V_\gamma V_\delta^* k \rangle = \langle V_\rho V_\sigma^* h, k \rangle = \langle V_\rho h, V_\sigma^* k \rangle = \delta_{\sigma(i, \ldots, j)} \delta_{\rho(j, \ldots, i)} \langle V_i h', V_j k' \rangle$. But, when $i \neq j$, $\langle V_i h', V_j k' \rangle = \langle h', V_j V_i k' \rangle = 0$ and, when $i = j$, $\langle V_i h', V_j k' \rangle = \langle V_i^r h', V_i k' \rangle = \langle h, V_i^{r+1} k' \rangle = \langle h, k \rangle = 0$, as required. □

Using Proposition 2.4, it is possible to give a complete classification up to conjugacy of the simplest shifts $\alpha$, for which $E_\alpha$ is one-dimensional.

Proposition 2.5. (i) Let $V$ be an isometry on a separable infinite-dimensional Hilbert space $H$ and let $\alpha$ be the $*$-endomorphism of $B(H)$ defined by $\alpha(A) = VAV^*$. Then $\alpha$ is a shift if and only if $V$ is either a pure isometry or $\cap_r V^* H$ is one-dimensional.

(ii) Let $\alpha, \beta$ be shifts on $B(H)$ defined by $\alpha(A) = VAV^*$ and $\beta(A) = WAW^*$, where $V$ and $W$ are isometries on $H$. Then $\alpha$ is conjugate to $\beta$ if and only if $V$ is unitarily equivalent to $\lambda W$ for some $\lambda \in S^1$ i.e. if and only if $V$ and $W$ have the same Wold decomposition.

Proof. (i) By Proposition 2.4, the conditions on $V$ are necessary for $\alpha$ to be a shift. Conversely, if $V$ satisfies the given conditions, then $V$ can be assumed to be a direct sum of copies of the unilateral shift and (possibly) a unitary map on a
one-dimensional subspace. Let \( \{e_i : i \in \mathbb{N}\} \) be the orthonormal basis associated with one of the shift summands. If \( T \in \alpha'(B(H)) \) then \( T = V^*S^*V \) for some \( S \in B(H) \) and hence \( T e_i = 0 \) for \( 1 \leq i \leq r \). It follows that, if \( T \in \bigcap_r \alpha'(B(H)) \) then the restriction of \( T \) to \( \bigcap_r V^*H \) is 0. The same argument applies to \( T^* \) and hence \( T \) leaves \( \bigcap_r V^*H \) globally invariant. Therefore \( \bigcap_r \alpha'(B(H)) = CP \) where \( P \) is the projection from \( H \) onto the zero or one-dimensional space \( \bigcap_r V^*H \).

(ii) This follows from Proposition 2.2. \( \square \)

The case in which \( E_\alpha \) has dimension one is very special. For example, in contrast to the result of Proposition 2.5(i), if \( E_\alpha \) has dimension greater than one, then each isometry \( V \) in \( E_\alpha \) is onto a space of infinite codimension and hence must be equivalent to an infinite direct sum of copies of the unilateral shift, possibly with an additional one-dimensional unitary summand.

Later in the paper an example will be produced to show that it may occur that \( \alpha(A) = \sum V_n A V_n^* = \sum W_n A W_n^* \) where all the \( V_n \) are pure and one of the \( W_n \) is not. It will also be shown that the condition that all the \( V_n \) are pure is not sufficient to ensure that \( \alpha \) is a shift. Both these examples rely on the notion of a product shift, that will now be introduced.

3. Product shifts

Let \( K \) be a Hilbert space and, for each \( i \in \mathbb{N}, \) let \( k_i \in K \) be a unit vector such that \( \sum |1 - \langle k_i, k_{i+1}\rangle| < \infty \). Recall that, as described in [3] or [6], the incomplete infinite tensor product \( \bigotimes^{k_1} K \) is the Hilbert space inductive limit of the sequence \( K \rightarrow K \otimes K \rightarrow K \otimes K \otimes K \rightarrow \cdots \), where the \( i \)th map takes \( h \) to \( h \otimes k_i \). Let \( \phi_i \) be the natural map from the \( i \)-fold tensor product \( K \otimes K \otimes \cdots \otimes K \) into \( \bigotimes^{k_i} K \) and let \( \psi_i \) be the corresponding map from \( K \otimes K \otimes \cdots \otimes K \) into \( \bigotimes^{k_{i+1}} K \). Then, by [3, Proposition 1.1], for each \( h \) in the \( r \)-fold tensor product \( K \otimes K \otimes \cdots \otimes K \), the sequence \( \phi_i(h), \phi_{r+1}(h \otimes k_{r+2}), \phi_{r+2}(h \otimes k_{r+2} \otimes k_{r+3}), \ldots \) has a limit in \( \bigotimes^{k_i} K \) and, by [3, Proposition 1.3], the map taking \( \psi_i(h) \) to this limit extends to an isomorphism \( \theta \) from \( \bigotimes^{k_i} K \) onto \( \bigotimes^{k_i} K \). Informally, we can describe \( \theta \) as taking \( \bigotimes x_i \) (where \( x_i = k_i \) for almost all \( i \)) to \( \bigotimes x_i \) (interpreting the last expression as the limit of \( \phi_1(x_1), \phi_2(x_1 \otimes x_2), \ldots \)).

**Proposition 3.1.** Let \( P \) be a projection on \( K; \) \( k_i \) be a sequence of unit vectors in \( K \) with \( \sum |1 - \langle k_i, k_{i+1}\rangle| < \infty \); and \( \theta \) be the isomorphism from \( \bigotimes^{k_{i+1}} K \) onto \( \bigotimes^{k_i} K \) taking \( \bigotimes x_i \) (where \( x_i = k_i \) for almost all \( i \)) to \( \bigotimes x_i \). Then the map \( \alpha_{k, \theta} \) defined on \( B(H) = B(\bigotimes^{k_i} K) \) by \( \alpha_{k, \theta}(T) = \theta(P \otimes T)\theta^{-1} \) is a shift.

**Proof.** The map \( \alpha_{k, \theta} \) is clearly a \(*\)-endomorphism. In the unital case, \( \alpha_{k, \theta}'(B(H))' = B(K \otimes \cdots \otimes K) \otimes \mathbb{C}1 \) (where the tensor product has \( r \) factors) and hence \( \bigcup \alpha_{k, \theta}'(B(H))' \) is weakly dense in \( B(H) \), from which it follows that \( \alpha_{k, \theta} \) is a shift. In the nonunital case let \( q \) be the greatest lower bound
of the projections \( \alpha_{k, p}^r(1) \). If \( q = 0 \) then, since \( \alpha_{k, p}^r(1)T = T \) for each \( T \in \bigcap_r \alpha_{k, p}^r(B(H)) \), \( T = qT = 0 \) for all such \( T \) and hence \( \alpha_{k, p}^r \) is a shift. If \( q \neq 0 \) then \( qa_{k, p}^r(1)(T)q = qa_{k, p}^r(T)q \) for all \( T \) because, as can be checked by induction, \( \alpha_{k, p}^r(1)\alpha_{k, p}^r(T)\alpha_{k, p}^r(1) = \alpha_{k, p}^r(T) \). Thus, since \( \alpha_{k, p}^r(q) = q \), \( a_{k, p}^r(qB(H)q) = qa_{k, p}^r(B(H))q = qa_{k, p}^r(1)(B(H))q \) and so, when both algebras are restricted to \( qH \), \( [\alpha_{k, p}^r(qB(H)q)]' = qa_{k, p}^r(1)(B(H))'q \). Hence, as in the unital case, \( \bigcap_r \alpha_{k, p}^r(qB(H)q) = \mathbb{C}q \); however, \( \bigcap_r \alpha_{k, p}^r(B(H)) \subseteq qB(H)q \) and, therefore, \( \bigcap_r \alpha_{k, p}^r(B(H)) = \mathbb{C}q \), as required. \( \square \)

If there exists a unitary \( U \) from a Hilbert space \( H \) onto an infinite tensor product \( \bigotimes_k K \) with the properties above, then any *-endomorphism \( \alpha \) of \( B(H) \) of the form \( (\text{Ad } U^*)\alpha_{k, p}(\text{Ad } U) \) will be called a product shift on \( B(H) \).

Not every shift on \( B(H) \) is a product as can be seen from Proposition 2.4. Any product shift \( \alpha_{k, p} \) has \( \alpha_{k, p}(1) \) of infinite codimension whereas if \( \alpha \) is the shift corresponding to a single unilateral shift \( V \) then \( \alpha(1) \) has codimension 1. Nevertheless the author is not aware of examples of nonproduct unital shifts on \( B(H) \): the following proposition shows that any sequence of isometries defining such a shift must be all pure.

**Proposition 3.2.** Let \( \alpha \) be a unital shift on \( B(H) \) defined by \( \alpha(A) = \sum_n V_n^*AV_n \) where \( V = V_1 \) is not pure. Then \( \alpha \) is a product shift.

**Proof.** Let \( h \) be a unit vector in \( \bigcap_r V^rH \) and note that, by Proposition 2.4, \( \bigcap_r V^rH \) is one-dimensional. It follows that \( Vh = \lambda h \) for some \( \lambda \in S^1 \) and, replacing \( V \) by \( \lambda V \), we can assume that \( Vh = h \).

Let \( c_{00} \) denote the space of functions \( f: \mathbb{N} \to \mathbb{N} \) such that \( f(r) = 1 \) for all sufficiently large \( r \) and, for each \( f \in c_{00} \), let \( b_f = \phi_s(V_f(1) \otimes \cdots \otimes V_f(s)) \in \bigotimes^s E_a \), where \( \phi_s \) is the natural map from the \( s \)-fold tensor product \( E_a \otimes \cdots \otimes E_a \) into \( \bigotimes^s E_a \) and where \( s \) is chosen sufficiently large so that \( f(r) = 1 \) for all \( r \geq s \). Then, by [6, Lemma 4.1.4], \( \{b_f: f \in c_{00}\} \) is an orthonormal basis for \( \bigotimes^s E_a \). Define \( \psi: \bigotimes^s E_a \to H \) extending \( \psi(b_f) = V_f(1) \cdots V_f(s)h \) (which is well defined since \( Vh = h \)). Then, for \( t \geq s \), \( \langle \psi(b_f), \psi(b_g) \rangle = \langle V_f(1) \cdots V_f(s)h, V_g(1) \cdots V_g(t)h \rangle = \delta_{f(1)g(1)} \cdots \delta_{f(s)g(s)} \delta_{f(s+1),g(s+1)} h, h \rangle = \delta_{f-g}(h, h) = \delta_{f-g}^a(h, h) \), so that \( \psi \) defines a unitary mapping from \( \bigotimes^s E_a \) onto a subspace \( K \) of \( H \).

For each \( \mu, \nu \) with \( |\mu| \geq 0 \) and \( |\nu| = r \geq 0 \), \( V_\mu V_\nu^*h = V_\mu V_\nu^*V^r h = \delta_{\nu(1, \ldots, 1)} V_\mu h \), so that \( K \) is equal to the closed linear span of the vectors \( V_\mu V_\nu^*h \) with \( |\mu| \geq 0 \) and \( |\nu| \geq 0 \). By Lemma 2.3 (i) it follows that the projection onto \( K \) belongs to \( \bigcap_r \alpha_{k, p}(B(H)) \) and hence, since \( \alpha \) is a unital shift, \( K = H \).

Note that
\[
(\psi^*V_n \psi)(\phi_r(V_f(1) \otimes \cdots \otimes V_f(r))) = \psi^*V_n V_f(1) \cdots V_f(r)h = \phi_{r+1}(V_n \otimes V_f(1) \otimes \cdots \otimes V_f(r)) ,
\]
from which it follows that
\[
(\psi^*V_n^*\psi)(\phi_r(V_{f(1)} \otimes \cdots \otimes V_{f(r)})) = (V_{f(1)}, V_n)\phi_{r-1}(V_{f(2)} \otimes \cdots \otimes V_{f(r)});
\]
hence
\[
(\text{Ad} \psi^*)\alpha(\text{Ad} \psi)(T_1 \otimes T_2 \otimes \cdots \otimes T_s \otimes 1) = \sum_n (\psi^*V_n\psi)(T_1 \otimes \cdots \otimes T_s \otimes 1)(\psi^*V_n\psi)^* = 1 \otimes T_1 \otimes \cdots \otimes T_s \otimes 1
\]
and thus that \( \alpha \) is a product shift. \( \square \)

It will now be shown how product shifts can be defined to illustrate the comments made after Proposition 2.5.

**Example 3.3.** Let \( K = \mathbb{C}^2 \); \( k_i = (1, 0) \) for each \( i \); and the isometries \( V_1, V_2, W_1, W_2 \) be defined on \( H = \bigotimes K \) by
\[
V_1(h) = (1/\sqrt{2}, 1/\sqrt{2}) \otimes h, \quad V_2(h) = (1/\sqrt{2}, -1/\sqrt{2}) \otimes h,
\]
\[
W_1(h) = (1, 0) \otimes h, \quad W_2(h) = (0, 1) \otimes h.
\]
Then \( \alpha_{k,1}(A) = V_1AV_1^* + V_2AV_2^* = W_1AW_1^* + W_2AW_2^* \) for each \( A \in B(H) \).

However, both \( V_1 \) and \( V_2 \) are pure isometries (as is \( W_2 \)) but \( \cap_n W_1^nH = \mathbb{C}k \), where \( k = \bigotimes (1, 0) \). \( \square \)

**Example 3.4.** Let \( H, V_1, V_2, \) and \( \alpha_{k,1} \) be as defined in Example 3.3 and let \( H \) be any Hilbert space. Then \( \alpha = \alpha_{k,1} \otimes \text{id} \) on \( B(H) \otimes B(H) \) is not a shift; however,
\[
\alpha(A) = (V_1 \otimes 1)A(V_1^* \otimes 1) + (V_2 \otimes 1)A(V_2^* \otimes 1)
\]
for each \( A \), and both \( V_1 \otimes 1 \) and \( V_2 \otimes 1 \) are pure isometries. Hence the converse of Proposition 2.4 is false. \( \square \)

The next main result gives a criterion for two unital product shifts to be conjugate. The proof is based on the following lemma:

**Lemma 3.5.** Let \( W : \bigotimes K \to \bigotimes K \) be a unitary for which \( (\text{Ad} \ W)\alpha_{k,1}(\text{Ad} \ W^*) = \alpha_{h,1} \). Then there exists a unitary \( V \) on \( K \) such that, for each \( r \geq 1 \) and each \( T_1, \ldots, T_r \in B(K) \),
\[
(\text{Ad} \ W)(T_1 \otimes \cdots \otimes T_r \otimes 1) = (VT_1V^* \otimes \cdots \otimes VT_rV^*) \otimes 1.
\]

**Proof.** For notational convenience, let \( H_h = \bigotimes K \) and \( H_k = \bigotimes K \). Then \( \alpha_{h,1}(B(H_h))' = \text{Ad}(W)[\alpha_{k,1}(B(H_k))'] \) and hence, for each \( T \in B(K) \),
\[
(\text{Ad} \ W)(T \otimes 1 \otimes 1 \otimes \cdots) = VT(V^* \otimes 1 \otimes 1 \otimes \cdots)
\]
for some unitary \( V \) on \( K \). Then, for each \( r \geq 0 \),
\[
(\text{Ad} \ W)\alpha_{k,1}(T \otimes 1 \otimes 1 \otimes \cdots) = \alpha_{h,1}(\text{Ad} \ W)(T \otimes 1 \otimes 1 \otimes \cdots)
\]
\[
= \alpha_{h,1}(VT(V^* \otimes 1 \otimes 1 \otimes \cdots)).
\]
Hence, for each $r \geq 0$, $(\text{Ad } W)(T_1 \otimes \cdots \otimes T_r \otimes 1) = (VT_1 V^*) \otimes \cdots \otimes (VT_r V^*) \otimes 1$. □

**Theorem 3.6.** The product shifts $\alpha_{k,1}$ and $\alpha_{h,1}$ are conjugate if and only if there exists a unitary map $V$ on $K$ with $\sum |1 - |(V^* k_i, h_i)|| < \infty$.

**Proof.** If $\sum |1 - |(V^* k_i, h_i)|| < \infty$ then, by [3, Proposition 1.3], the map taking $\bigotimes x_i$ (where $x_i = k_i$ for almost all $i$) to $\bigotimes x_i$ extends to an isomorphism from $\bigotimes^{k_i} K$ onto $\bigotimes^{h_i} K$, where $\alpha_i$ is the complex number of unit modulus such that $|\langle k_i, V^* h_i \rangle| = |\langle k_i, \alpha_i V^* h_i \rangle|$. However there is an isomorphism from $\bigotimes^{h_i} K$ onto $\bigotimes^{h_i} K$ taking $\bigotimes x_i$ to $\bigotimes \alpha_i V x_i$ and hence there is an isomorphism $W$ from $\bigotimes^{h_i} K$ onto $\bigotimes^{h_i} K$ taking $\bigotimes x_i$ (where $x_i = k_i$ for almost all $i$) to $\bigotimes \alpha_i V x_i$. Then, for $T_1, \ldots, T_r \in B(K)$,

$$(\text{Ad } W)\alpha_{k,1}(\text{Ad } W^*)(T_1 \otimes \cdots \otimes T_r \otimes 1) = \alpha_{h,1}(T_1 \otimes \cdots \otimes T_r \otimes 1),$$

from which it follows, by linearity and continuity that $\alpha_{k,1}$ is conjugate to $\alpha_{h,1}$.

Conversely, if $\alpha_{k,1}$ is conjugate to $\alpha_{h,1}$ then, by Lemma 3.5, there exists an isomorphism $W$ from $\bigotimes^{k_i} K$ onto $\bigotimes^{h_i} K$ and a unitary $V$ on $K$ such that $(\text{Ad } W)(T_1 \otimes \cdots \otimes T_r \otimes 1) = (VT_1 V^*) \otimes \cdots \otimes (VT_r V^*) \otimes 1$ for all $T_1, \ldots, T_r \in B(K)$. In particular, this equation holds when $T_i$ is the one-dimensional projection onto the subspace spanned by $k_i$. In this case, the decreasing sequence of projections $T_1 \otimes \cdots \otimes T_r \otimes 1$ is weakly convergent as $r \to \infty$; by considering the effect of $T_1 \otimes \cdots \otimes T_r \otimes 1$ on an orthonormal basis $\{e_{i_1} \otimes e_{i_2} \otimes \cdots : i_1, i_2, \ldots \in \mathbb{N}\}$ (where $e_{i_r} = k_r$ for each $r$ and $i_k = 1$ for all sufficiently large $k$) it can be seen that the weak limit is the one-dimensional projection onto the subspace spanned by $k = \bigotimes k_i$. Hence the projections $(VT_1 V^*) \otimes \cdots \otimes (VT_r V^*) \otimes 1$ are weakly convergent to a one-dimensional projection in $\bigotimes^{h_i} K$; let $x$ be a unit vector in the corresponding one-dimensional subspace. If $\phi_r$ denotes the natural map from the $r$-fold tensor product $K \otimes \cdots \otimes K$ into $\bigotimes^{h_i} K$ then there exists $s \in \mathbb{N}$ and a unit vector $x_s$ in $\phi_s(K \otimes \cdots \otimes K)$ such that $\|x - x_s\| < \frac{1}{2}$. Then, for each $r$,

$$\|[(VT_1 V^*) \otimes \cdots \otimes (VT_r V^*) \otimes 1]x_s\| \geq \|x\| - \|x - [(VT_1 V^*) \otimes \cdots \otimes (VT_r V^*) \otimes 1]x_s\| > \frac{1}{2}.$$  

It follows that $\prod_{i=1}^{s+1} |(k_i, V^* h_i)| \to 0$ as $r \to \infty$ and hence, by Lemma 2.4.1 (II) of [6], $\sum |1 - |(k_i, V^* h_i)|| \to < \infty$, as required. □

Theorem 3.6 can be used to produce examples of nonconjugate but outer-conjugate unital shifts. Let $\log^k$ denote the $k$-fold composite $\log \circ \cdots \circ \log$ and, for each $r \in \mathbb{N}$, define a sequence $f_r$ by

$$f_r(n) = \begin{cases} 1 & \text{if } \log^r(n) \leq 0, \\ 1/(n \log(n) \log^2(n) \cdots \log^r(n)) & \text{if } \log^r(n) > 0. \end{cases}$$
Let $K$ be a separable Hilbert space (of dimension at least 2) with an orthonormal basis $\{e_i\}$ and, for each $n \in \mathbb{N}$, define $k_{r,n} \in K$ by $k_{r,n} = (1 - f_r(n))e_1 + [f_r(n)(2 - f_r(n))]^{1/2}e_2$.

**Lemma 3.7.** (i) $\|k_{r,n}\| = 1$ for each $r$ and $n$.

(ii) $\sum_n |1 - \langle k_{r,n}, k_{r,n+1}\rangle| < \infty$ for each $r$.

**Proof.** Claim (i) is easily verified. To check claim (ii), note that

$$o < \langle k_{r,n}, k_{r,n+1}\rangle < \|k_{r,n}\| \|k_{r,n+1}\| = 1$$

and hence that

$$0 \leq |1 - \langle k_{r,n}, k_{r,n+1}\rangle|$$

$$= f_r(n) + f_r(n + 1) - f_r(n)f_r(n + 1)$$

$$- [f_r(n)f_r(n + 1)(2 - f_r(n))(2 - f_r(n + 1))]^{1/2}$$

$$\leq 2f_r(n) - f_r(n)f_r(n + 1)(2 - f_r(n))$$

$$= 2(f_r(n) - f_r(n + 1))$$

However $\sum_n 2(f_r(n) - f_r(n + 1))$ is convergent, from which the result follows. \(\Box\)

**Proposition 3.8.** For each $r \in \mathbb{N}$, let $\alpha_r$ be the product shift $\alpha_{k_{r,1}}$ on $\bigotimes_{n=1}^\infty k_{r,n}K$, where $k_{r,n}$ is defined above. Then, if $r \neq s$, $\alpha_r$ is outer conjugate but not conjugate to $\alpha_s$.

**Proof.** For definiteness suppose that $r > s$. By construction $\alpha_r$ and $\alpha_s$ have the same multiplicity $\dim(K)$ and hence are outer conjugate by [7, Theorem 2.4]. If they are conjugate then, by Theorem 3.6, there exists a unitary $V$ on $K$ such that $\sum_n |1 - (Vk_{r,n}k_{s,n})| |< \infty$. It follows that $|\langle Vk_{r,n}, k_{s,n}\rangle| \to 1$ as $n \to \infty$ and hence that $|\langle Ve_1, e_1\rangle| = 1$, so that $Ve_1 = \alpha_1 e_1$ for some $\alpha \in S^1$. Hence,

$$|\langle Vk_{r,n}, k_{s,n}\rangle| = |(1 - f_s(n))(1 - f_s(n))(\alpha e_1, e_1)$$

$$+ [f_s(n)f_s(n)(2 - f_s(n))(2 - f_s(n))]^{1/2}(Ve_2, e_2)|$$

$$\leq (1 - f_s(n)(1 - f_s(n)) + 2f_s(n)^{1/2}f_s(n)^{1/2}$$

$$= 1 - f_s(n)[1 - f_s(n) + f_s(n)f_s(n)^{-1} - 2f_s(n)^{1/2}f_s(n)^{-1/2}]$$

However, for all sufficiently large $n$, $f_r(n)f_s(n)^{-1} = 1/(\log^{s+1}(n) \cdot \cdot \cdot \log^r(n)) \to 0$ as $n \to \infty$ and therefore, for all sufficiently large $n$, $|\langle Vk_{r,n}, k_{s,n}\rangle| \leq 1 - f_s(n)/2$ and therefore, $\sum |1 - |\langle Vk_{r,n}, k_{s,n}\rangle||$ is not convergent, giving a contradiction. \(\Box\)

**References**


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