ON INTERSECTION OF COMPACTA IN EUCLIDEAN SPACE II

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Abstract. Suppose that $X$ is a compact subset of $n$-dimensional Euclidean space $\mathbb{R}^n$. If every map $f: Y \to \mathbb{R}^n$ of a compactum $Y$ can be approximated by a map avoiding $X$ then $\dim X \times Y < n$.

0. Introduction

The main result of this paper is the following Theorem 1 which is the inverse of the main result of [D1].

Theorem 1. Let $X$ be a compactum in Euclidean space $\mathbb{R}^n$ and suppose that for a compact metric space $Y$, the space of maps $C(Y, \mathbb{R}^n - X)$ is dense in $C(Y, \mathbb{R}^n)$. Then $\dim X \times Y < n$.

Here $C(X, Z)$ denotes the space of continuous maps between $X$ and $Z$ with the compact-open topology.

This theorem plus Theorem 1 from [D1] implies the following:

Theorem 2. For an arbitrary compactum $Y$ and for a codimension three tame compact subset $X \subset \mathbb{R}^n$ the following are equivalent:

$(a)$ $C(Y, \mathbb{R}^n - X)$ is dense in $C(Y, \mathbb{R}^n)$.

$(b)$ $\dim X \times Y < n$.

Remark 1 [D2]. There exist compacta $X, Y$ with $\dim X = \dim Y = n - 2$ and $\dim X \times Y < n$ for arbitrary $n$.

The proof of Theorem 1 is based on Spanier-Whitehead duality and some elements of extension theory. The top of the preliminary work in extension theory is

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theory is Theorem 3 which is a generalization of the inequality \( \dim \geq \dim_S \), where \( \dim_S \) is the stable cohomotopy dimension [D3].

1. Extension theory

Notation [Ku]. Let \( Y \) and \( M \) be topological spaces; then

\[ Y \times M \iff \text{for arbitrary closed subset } A \subset Y \text{ and arbitrary continuous map } \varphi: A \to M \text{ there exists a continuous extension } \widetilde{\varphi}: Y \to M. \]

We will consider only the case when \( M \) is a \( CW \)-complex.

**Proposition 1.** Let \( A \) be a closed subset of \( Y \). Then the property \( Y \times M \) implies the property \( A \times M \).

**Proposition 2.** Let \( U \) be an open subset of a metric space \( Y \). Then \( Y \times M \) implies \( U \times M \).

**Proof.** There exists a filtration \( F_1 \subset F_2 \subset \cdots \subset F_i \subset \cdots \) of \( U \), where \( F_i = Y - O_{1/i}(Y - U) \), where \( O_e(B) \) denotes the open \( e \)-neighborhood of \( B \) in \( Y \). For arbitrary \( \varphi: A \to M \) by induction construct a sequence of maps \( \varphi_i: A \cup F_i \to M \) with the property \( \varphi_{i+1}|_{A \cup F_i} = \varphi_i \) and \( \varphi_i|_A = \varphi \) for all \( i \). The union \( \bigcup \varphi_i \) is a continuous extension of the map \( \varphi \) to \( U \).

**Proposition 3.** Let \( B \) be a closed subset in \( Y \). Then the properties \( B \times M \) and \( (Y - B) \times M \) imply \( Y \times M \).

**Proof.** Suppose that \( \varphi: A \to M \) is an arbitrary map and \( A \) is an arbitrary closed subset in \( Y \). Due to the property \( B \times M \) there exists an extension \( \varphi': A \cup B \to M \). Since \( M \in ANE \) there exists an extension \( \bar{\varphi}: O \to M \) to an open set \( O \supset A \cup B \). Let \( W \) be an open set with \( A \cup B \subset W \subset [W] \subset O \) where \([W]\) is the closure of \( W \). Apply \( Y - B \times M \) to extend the map \( \bar{\varphi}|_{\partial W} \) to the map \( \tilde{\varphi}: Y - W \to M \). The union \( \tilde{\varphi} \cup \bar{\varphi}|_{[W]} \) is a continuous map \( \psi: Y \to M \) with the restriction \( \psi|_A = \varphi \).

**Proposition 4.** Suppose that compact \( Y \) is a union \( \bigcup_{i=1}^m F_i \) of closed subsets. Then the properties \( F_i \times M \) imply \( Y \times M \).

**Proof.** By induction on \( m \) and using Proposition 1.

Let \( \Sigma M \) denote the suspension of \( M \).

**Lemma 1.** The property \( X \times M \) implies \( (X \times [0,1]) \times \Sigma M \) for metric spaces \( X \).

**Proof.** Define an open set \( V(r, U) = \{(x, t) \in X \times [0,1] | x \in U \} \) and \( r - d(x, X - U) < t < r + d(x, X - U) \) where \( U \) is an open subset in \( X \) and \( d \) is the metric on \( X \). Then for every \( V(r, U) \) the boundary \( \partial V(r, U) \) in \( X \times R \) is equal to the union \( F_1 \cup F_2 \) where

\[ F_1 = \{(x, t_1): t = \max\{0, r - d(x, X - U)\} \}
\]

and

\[ F_2 = \{(x, t): t = \min\{1, r + d(x, X - U)\} \}.
\]
It is easy to see that \( F_{i}, i = 1, 2 \) is homeomorphic to the closure \([U]\) of \( U \). Since \( F_{i} \subset M \) by Proposition 4 we have \( \partial V_{x}M \).

Let \( A \subset X \times [0, 1] \) be a closed subset and \( \varphi: A \to \Sigma M \) be a continuous map. The suspension \( \Sigma M \) consists of the union of two cones: \( \Sigma M = \text{con}^{+} M \cup \text{con}^{-} M \) with the vertices \( x^{+} \) and \( x^{-} \). Denote \( \varphi^{-1}(x^{+}) = A^{+} \) and \( \varphi^{-1}(x^{-}) = A^{-} \). Since the sets \( V(r, U) \) generate a basis of the topology on \( X \times [0, 1] \) there exists an open set \( V \) such that \( A^{+} \subset V, A^{-} \subset X \times [0, 1] - [V] \) and \( V = \bigcup_{i=1}^{m} V(r_{i}, u_{i}) \). Since \( \partial V \subset \bigcup_{i=1}^{m} \partial V(r_{i}, u_{i}) \), Proposition 1 and 4 imply that \( \partial V_{x}M \). Since \( \Sigma M \) is homeomorphic to \( M \times \mathbb{R} \) there exists an extension \( \varphi^{'}: \partial V \to \Sigma M \). Let \( A_{-} = \varphi^{'}(x^{-}) \) and \( A_{+} = \varphi^{'}(x^{+}) \). Since spaces \( \text{con}^{+} A_{-} \) and \( \text{con}^{-} A_{-} \) are contractable there are extensions \( \psi^{+} \) and \( \psi^{-} \). Consider the restrictions \( \psi^{+} = \psi|_{(A \cap [V]) \cup \partial V} \) and \( \psi^{-} = \psi|_{(A - V) \cup \partial V} \). Since spaces \( \text{con}^{+} M \) and \( \text{con}^{-} M \) are contractable there are extensions \( \psi^{+} \) and \( \psi^{-} \). The union \( \psi^{+} \cup \psi^{-} \) gives the map \( \psi: X \times [0, 1] \to \Sigma M \) which is an extension of \( \varphi \).

**Proposition 5.** The property \( X \tau M \) implies \( \text{con} X \tau \Sigma M \).

**Proof.** By virtue of Lemma 1, \( X \times [0, 1] \tau \Sigma M \). By Proposition 2 we have \( X \times [0, 1] \tau \Sigma M \). Since \( pt \tau \Sigma M \) and \( \psi - pt \approx X \times [0, 1] \) Proposition 3 implies \( \text{con} X \tau \Sigma M \).

The following proposition has a similar proof.

**Proposition 6.** The property \( X \tau M \) implies \( \Sigma X \tau \Sigma M \).

**Corollary.** \( X \tau M \) implies \( \Sigma X \tau \Sigma M \).

**Theorem 3.** For arbitrary CW-complex \( M \) and compactum \( X \) the property \( X \tau M \) implies the property \( X \tau \Omega_{i}^{i} M \) for any \( i = 1, 2, \ldots, \infty \).

Here \( \Omega_{i}^{i} Y \) denotes iterated loop space.

**Proof.** Let \( i < \infty \). Consider the diagram

\[
\begin{array}{ccc}
[S^{i} A, \Sigma M] & \longrightarrow & [A, \Omega_{i}^{i} \Sigma M] \\
\uparrow & & \uparrow \\
[S^{i} X, \Sigma M] & \longrightarrow & [X, \Omega_{i}^{i} \Sigma M].
\end{array}
\]

All horizontal arrows are isomorphisms, the left vertical arrow is an epimorphism due to the corollary of Proposition 6. Thus the right vertical arrow is an epimorphism too. Hence \( X \tau \Omega_{i}^{i} M \).

Recall that \( \Omega_{i}^{i} \Sigma_{i} \Omega_{i}^{i} M = \lim \Omega_{i}^{i} \Sigma_{i} M \). Then compactness of \( X \) and the properties \( X \tau \Omega_{i}^{i} M \) for \( i < \infty \) imply the property \( X \tau \Omega_{i}^{i} \Sigma_{i} M \).

**Remark 2.** \( X \tau \Omega^{\infty} \Sigma^{\infty} S^{n} \iff \dim_{S} X \leq n \) where \( \dim_{S} \) is the stable cohomotopy dimension [D3]. So, for \( M = S^{n} \), Theorem 3 claims the inequality \( \dim X \geq \dim_{S} X \).
Problem. Does $X \tau M$ imply $X \tau S P^\infty M$, where $S P^\infty$ is the infinite symmetric power?

2. Spanier-Whitehead duality

Lemma 2. Let $U$ be an open $n$-dimensional ball and $X \subset U$ be a closed subset. Let $M = U - X$, and let $X'$ be the one-point compactification of $X$. Then for a finite-dimensional compactum $Y$ and for large enough $m$ there is an isomorphism $\beta_Y : [Y, \Omega^m \Sigma^m M] \to [\Sigma^m(Y \wedge X'), S^{m+n-1}]$ which depends naturally on $Y$.

A proof of Lemma 2 actually is contained in [D4] (see the lemma) and it is a consequence of Spanier-Whitehead duality.

Consider the one-point compactification $U'$ of $U$. In the $n$-dimensional sphere $U' \approx S^n$ choose a decreasing sequence $\{K_i\}$ of polyhedra with intersection $\bigcap K_i = X'$. Spanier-Whitehead duality claims that for finite-dimensional $Y$ and large enough $m$ there is an isomorphism $\beta_i : [\Sigma^m Y, \Sigma^m (U' - K_i)] \to [\Sigma^m(Y \wedge K_i), S^{m+n-1}]$. The limit $\lim_{i \to \infty} \beta_i$ gives the isomorphism $\beta_Y$. This conclusion is based on the following propositions.

Proposition 7. Let the compactum $X$ be a limit space of an inverse system $\{X_i, p_i^{i+1}\}$ of compacta and let $M$ be a CW-complex. Then $[X, M] = \varinjlim [X_i, M]$.

Proposition 8. Suppose $M$ is a limit space of a direct system $\{M_i, \varphi_i^{i+1}\}$ of CW-complexes and inclusions. Then for any compactum $Y$ there is an equality $[Y, M] = \varprojlim [Y, M_i]$.

Lemma 3. Let $X$ be a compact subset of $\mathbb{R}^n$, then for compactum $Y$ the following are equivalent:

(1) the space $C(Y, \mathbb{R}^n - X)$ is dense in $C(Y, \mathbb{R}^n)$,

(2) for any open ball $U \subset \mathbb{R}^n$, $Y \tau M_U$, where $M_U = U - X$.

Proof. (1) $\Rightarrow$ (2). Let $A$ be a closed subset of $Y$ and $\varphi : A \to M_U$ be an arbitrary map. Choose an arbitrary extension $\varphi' : Y \to U$. There exists $\epsilon > 0$ such that every map $\psi \epsilon$-close to $\varphi$ is homotopic to $\psi$ in $M_U$. Approximate $\varphi'$ by $\psi : Y \to \mathbb{R}^n - X$ such that $\psi(Y) \subset M_U$ and $\psi$ is $\epsilon$-close to $\varphi'$. The homotopy extension theorem implies that there is an extension $\overline{\varphi} : Y \to M_U$ of $\varphi$.

(2) $\Rightarrow$ (1) See the proof of Theorem 1 in [D1].

3. Proof of Theorem 1

Suppose the contrary: $\dim X \times Y \geq n$. We can assume that $\dim X \times Y = n$. (Otherwise consider $X = X \times \{0\} \subset \mathbb{R}^n \times \mathbb{R}^{m-n} = \mathbb{R}^m$ where $m = \dim X \times Y$. It is easy to see that $C(Y, \mathbb{R}^m - X)$ is dense in $C(Y, \mathbb{R}^m)$. The case of $\dim Y =
∞ is excluded because if dim Y ≥ n then always there is an essential map of Y onto n-dimensional cube.)

Let Z be a cone over Y. Lemma 3 implies the property Y ∩ M_U for an arbitrary open ball U ⊂ \( \mathbb{R}^n \). By virtue of Proposition 5 we have Z ∩ Σ M_U for every open ball U ⊂ \( \mathbb{R}^n \). Denote \( W = U \cap X \) and let \( W' \) be the one-point compactification of W. Let V be an arbitrary open subset of Z and A = Z - V.

The set U is naturally embedded in Σ U as the equator. Then Σ U - W is homotopy equivalent to Σ M_U. By Lemma 2 there is the following diagram

\[
\begin{align*}
[A, \Omega^k \Sigma^k \Sigma M_U] & \xrightarrow{\beta_1} [W' \wedge A, \Omega^k \Sigma^k S^n] \\
\uparrow \alpha_1 & \uparrow \alpha_2 \\
[Z, \Omega^k \Sigma^k \Sigma M_U] & \xrightarrow{\gamma_1} [W' \wedge Z, \Omega^k \Sigma^k S^n],
\end{align*}
\]

where for large enough k all horizontal arrows are isomorphisms. Theorem 3 implies that \( \alpha_1 \) is an epimorphism. Therefore \( \alpha_2 \) is an epimorphism.

Since dim(\( Z \wedge W' \)) = n + 1 < 2n - 1 the homomorphisms \( \beta_2, \gamma_2 \) in the following diagram are isomorphisms:

\[
\begin{align*}
[W' \wedge A, \Omega^k \Sigma^k S^n] & \xrightarrow{\beta_2} [W' \wedge A, S^n] \\
\uparrow \alpha_2 & \uparrow \alpha_3 \\
[W' \wedge Z, \Omega^k \Sigma^k S^n] & \xrightarrow{\gamma_2} [W' \wedge Z, S^n].
\end{align*}
\]

Hence \( \alpha_3 \) is an epimorphism.

Since dim \( W' \times Z = n + 1 \) then for homomorphism \( \beta_3, \gamma_3 \) in the diagram

\[
\begin{align*}
[W' \wedge A, S^n] & \xrightarrow{\beta_3} [W' \wedge A, K(Z, n)] \\
\uparrow \alpha_3 & \uparrow \alpha_4 \\
[W' \wedge Z, S^n] & \xrightarrow{\gamma_3} [W' \wedge Z, K(Z, n)],
\end{align*}
\]

are epimorphisms. Therefore \( \alpha_4 \) is an epimorphism. Here \( K(Z, n) \) is the Eilenberg-MacLane complex.

Consider the cohomology exact sequence of the pair \( (W' \wedge Z, W' \wedge A) : \cdots \leftarrow \tilde{H}^{n+1}(W' \wedge Z) \leftarrow \tilde{H}^n(W' \wedge Z, W' \wedge A) \leftarrow \tilde{H}^n(W' \wedge A) \xrightarrow{\alpha_4} \tilde{H}^n(W' \wedge Z) \). Since Z is contractible then \( H^*(W' \wedge Z) = 0 \). This and the fact that \( \alpha_4 \) is an epimorphism imply that \( \tilde{H}^{n+1}(W' \wedge Z, W' \wedge A) = 0 \). Because \( (W' \wedge Z)/(W' \wedge A) = (W' \times Z)/(W' \times A) \cup (\ast) \times Z \) (where \( \ast = W' - W \) ) then \( \tilde{H}^{n+1}(W' \times Z, W' \times Z) \cup (\ast) \times Z) = 0 \). In other terms \( H_c^{n+1}(W \times V) = 0 \). Since the sets of the form \( W \times V \) are a basis in \( X \times Z \) we have \( [Kuz] \) the inequality \( c - \text{dim}_Z(X \times Z) < n + 1 \). Since both \( X \) and \( Y \) are finite-dimensional then \( \text{dim}(X \times \text{con} Y) < n + 1 \). We have reached a contradiction.

**References**


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