PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 113, Number 4, December 1991

THE ACTION OF THE STEENROD SQUARES ON THE MODULAR INVARIANTS OF LINEAR GROUPS

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(Communicated by Frederick R. Cohen)

Dedicated to Professor Frank Peterson on his sixtieth birthday

ABSTRACT. We compute the action of the Steenrod squares on the Dickson invariants of the group $GL_n = GL(n, \mathbf{Z}/2)$ and the Mùi invariants of the subgroup T_n consisting of all upper triangular matrices with 1 on the main diagonal. Our method is very elementary. Roughly speaking, we read off the above action from the expansion of the Mùi invariants in terms of Dickson and Mùi invariants of fewer variables.

1. INTRODUCTION

Let $GL_n = GL(n, \mathbf{Z}/2)$ and let T_n be the Sylow 2-subgroup of GL_n consisting of all upper triangular matrices with 1 on the main diagonal. These two groups act on $\mathbf{Z}/2[y_1, \dots, y_n]$ in the usual manner.

Dickson and Mùi, respectively, computed the invariant rings

$$\mathbf{Z}/2[y_1, \dots, y_n]^{GL_n} \quad \text{and} \quad \mathbf{Z}/2[y_1, \dots, y_n]^{T_n}$$

respectively, as follows.

In [3] Mùi defined the invariant V_n by

$$V_n = V_n(y_1, \dots, y_n) = \prod_{\lambda_i \in \mathbf{Z}/2} (\lambda_1 y_1 + \cdots + \lambda_{n-1} y_{n-1} + y_n).$$

The Dickson invariant $Q_{n,s} = Q_{n,s}(y_1, \dots, y_n)$, for $0 \le s < n$, can be defined by the inductive formula

$$Q_{n,s} = Q_{n-1,s-1}^2 + V_n Q_{n-1,s}.$$

Here, by convention, $Q_{n,n} = 1$ for any n, and $Q_{1,0} = y_1$.

Dickson proved in [2] that

$$\mathbf{Z}/2[y_1, \dots, y_n]^{GL_n} = \mathbf{Z}/2[Q_{n,0}, \dots, Q_{n,n-1}].$$

Received by the editors April 5, 1990; the contents of this paper were presented in November 1989 to a seminar at MSRI, Berkeley.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 55S05.

Key words and phrases. Modular invariants, Steenrod operations.

Research at MSRI supported in part by NSF Grant DMS-8505550.

0002-9939/91 $1.00 + $.25 per page

Mùi showed in [3] that

$$\mathbf{Z}/2[y_1, \dots, y_n]^{T_n} = \mathbf{Z}/2[V_1, \dots, V_n].$$

Let us consider the elementary abelian 2-group $E^n \cong (\mathbf{Z}/2)^n$. As is well known $H^*(BE^n) \cong \mathbf{Z}/2[y_1, \dots, y_n]$ with $\deg y_i = 1$. Here and in what follows, the cohomology is always taken with coefficients in $\mathbf{Z}/2$. The Dickson and Mùi invariants have successfully been used in algebraic topology by many authors since Mùi's work [3], where he proved, among other things, that

$$\mathbf{Z}/2[y_1, \dots, y_n]^{GL_n} = \mathrm{Im}[\mathrm{Res}_1 \colon H^*(B\Sigma_{2^n}) \to H^*(BE^n)],$$
$$\mathbf{Z}/2[y_1, \dots, y_n]^{T_n} = \mathrm{Im}[\mathrm{Res}_2 \colon H^*(B\Sigma_{2^n,2}) \to H^*(BE^n)].$$

Here Σ_m denotes the symmetric group on m letters, $\Sigma_{m,2}$ is its Sylow 2-subgroup, and Res_1, Res_2 are the restriction homomorphisms induced by the regular permutation representation $E^n \subset \Sigma_{2^n,2} \subset \Sigma_{2^n}$ of E^n.

The mod 2 Steenrod algebra A acts on $H^*(BE^n) = \mathbf{Z}/2[y_1, \dots, y_n]$ by means of the Cartan formula together with the relations

$$Sq^0 y_i = y_i, \qquad Sq^1 y_i = y_i^2, \qquad Sq^j y_i = 0 \quad \text{for } 1 < j \text{ and } 1 \le i \le n.$$

Since this action commutes with the actions of GL_n and T_n, so it induces a natural action of A on $\mathbf{Z}/2[y_1, \dots, y_n]^G$ for $G = GL_n$ or T_n. Note that, $\deg V_n = 2^{n-1}$ and $\deg Q_{n,s} = 2^n - 2^s$ because $\deg y_i = 1$.

In this paper the action of the Steenrod squares on the Dickson and Mùi invariants is explicitly determined by use of a very elementary method. We prove

Theorem A.

$$Sq^i V_n = \begin{cases} V_n, & i = 0, \\ V_n Q_{n-1,s}, & i = 2^{n-1} - 2^s,\ 0 \le s < n-1, \\ V_n^2, & i = 2^{n-1}, \\ 0, & \text{otherwise.} \end{cases}$$

We also have the following result, which was first obtained in joint work with Nguyên N. Hai [11] by another method.

Theorem B (Hai and Hung [11]).

$$Sq^i Q_{n,s} = \begin{cases} Q_{n,r}, & i = 2^s - 2^r,\ r \le s, \\ Q_{n,r} Q_{n,t}, & i = 2^n - 2^t + 2^s - 2^r,\ r \le s < t, \\ Q_{n,s}^2, & i = 2^n - 2^s, \\ 0, & \text{otherwise.} \end{cases}$$

Many authors have studied the above action. However, their results can be divided into two kinds: Either they are only valid for i a power of 2, or they are given only by inductive procedures. (See Singer [12] and Campbell [1] for results concerning Theorem A, and Madsen [5], Madsen-Milgram [6, Chapter 3], Smith-Switzer [14], and Wilkerson [16] for results related to Theorem B. See also May [9, §I.3], Mann [7] and Mann-Milgram [8] for results related to Theorem B with the coefficient ring $\mathbf{Z}/p$ for $p > 2$.) Also by a direct computation, Singer obtained Theorem B for $n = 2, 3$ in [13].

It should be noted that the method which we present in this paper is very elementary. Roughly speaking, we read off the action of any Steenrod operation on the Dickson and Mùi invariants from the expansion of the Mùi invariants in terms of Dickson and Mùi invariants of fewer variables (see Lemmas 3.1 and 4.1 for details).

2. Preliminaries

Let ξ_i be the Milnor element of dimension $2^i - 1$ in the dual algebra A_* of the mod 2 Steenrod algebra A. Milnor showed in [10] that

$$A_* \cong \mathbf{Z}/2[\xi_1, \xi_2, \dots]$$

as algebras.

Given a sequence of nonnegative integers $R = (r_1, \dots, r_m)$, we denote by $St^R \in A$ the dual of $\xi_1^{r_1} \cdots \xi_m^{r_m}$ with respect to the Milnor basis of A_* consisting of all monomials in the ξ_i's. In particular, for $R = (r)$, $St^{(r)} = Sq^r$. Madsen-Milgram [6] and Mùi [4] have described St^R in terms of Dickson invariants by means of the homomorphism $d_m^* P_m$, a generalization of $d^* P$, which was first studied by Steenrod [15].

Suppose X is a topological space. Let

$$P_m \colon H^i(X) \to H^{2^m i}(E\Sigma_{2^m} \times_{\Sigma_{2^m}} X^{2^m})$$

be the Steenrod power map, which sends u to $1 \otimes u^{2^m}$ at cochain level.

The inclusion $E^m \subset \Sigma_{2^m}$ mentioned in the introduction, together with the diagonal map of X and the Künneth formula, induces the homomorphism

$$d_m^* \colon H^*(E\Sigma_{2^m} \times_{\Sigma_{2^m}} X^{2^m}) \to H^*(BE^m) \otimes H^*(X).$$

As is well known [3], the Weyl group of E^m in Σ_{2^m} is isomorphic to GL_m. Under the identification $H^*(BE^m) = \mathbf{Z}/2[x_1, \dots, x_m]$, the Weyl group GL_m acts on $\mathbf{Z}/2[x_1, \dots, x_m]$ as usual. A classical result asserts that

$$\operatorname{Im}(d_m^*) \subset \mathbf{Z}/2[x_1, \dots, x_m]^{GL_m} \otimes H^*(X).$$

Now St^R can be described as follows.

2.1. **Theorem** (Madsen-Milgram [6, Chapter 3], Mùi [4]).

$$d_m^* P_m(x) = \sum_{R=(r_1, \dots, r_m)} Q_{m,0}^{q-(r_1+\cdots+r_m)} Q_{m,1}^{r_1} \cdots Q_{m,m-1}^{r_{m-1}} \otimes St^R(x)$$

for any $x \in H^q(X)$. *Here* $Q_{m,s} = Q_{m,s}(x_1, \dots, x_m)$ *for* $0 \le s < m$.

For $m = 1$, $d_1^* P_1$ is nothing but $d^* P$ of Steenrod [15]. Mùi also observes that

2.2. **Proposition** [3, 4]. $d_m^* P_m$ *is a natural homomorphism preserving cup product and satisfying*

(i) $d_m^* P_m = d^* P d_{m-1}^* P_{m-1}^*$,
(ii) $d_m^* P_m(x) = V_{m+1}(x_1, \cdots, x_m, x)$.

Here x *is the generator of* $H^*(B\mathbf{Z}/2) \cong \mathbf{Z}/2[x]$ *with* $\deg x = 1$.

3. Proof of Theorem A

3.1. **Lemma.** *There exists uniquely an expansion*

$$V_{m+n}(x_1, \dots, x_m, y_1, \dots, y_n) = \sum_{R=(r_1, \dots, r_m)} Q_{m,0}^{2^{n-1}-(r_1+\cdots+r_m)} Q_{m,1}^{r_1} \cdots Q_{m,m-1}^{r_{m-1}} \cdot \varphi^R(V_1, \dots, V_n),$$

with $\varphi^R \in \mathbf{Z}/2[V_1, \dots, V_n]$, $Q_{m,s} = Q_{m,s}(x_1, \dots, x_m)$ *for* $0 \le s < m$, *and* $V_r = V_r(y_1, \dots, y_r)$ *for* $1 \le r \le n$. *Furthermore*

$$St^R(V_n) = \varphi^R(V_1, \dots, V_n).$$

Proof. From Proposition 2.2, we have

$$d_m^* P_m V_n(y_1, \dots, y_n) = V_{m+n}(x_1, \dots, x_m, y_1, \dots, y_n).$$

So, according to Theorem 2.1, there exists such an expansion. The uniqueness of the expansion follows from the algebraic independence of $V_1, \dots, V_n$ over $\mathbf{Z}/2[x_1, \dots, x_m]$.

Finally, we obtain from another application of Theorem 2.1 the last equation of the lemma.

3.2. *Proof of Theorem* A. We now apply Lemma 3.1 with $m = 1$. Note that, in this case $St^{(r)}$ is nothing but the Steenrod operation Sq^r.

$$\begin{aligned}
d^* PV_n(y_1, \dots, y_n) &= V_{n+1}(x, y_1, \dots, y_n) \\
&= \prod_{\lambda, \lambda_i} (\lambda x + \lambda_1 y_1 + \dots + \lambda_{n-1} y_{n-1} + y_n) \\
&= \prod_{\lambda_i} (\lambda_1 y_1 + \dots + \lambda_{n-1} y_{n-1} + y_n) \prod_{\lambda_i} (\lambda_1 y_1 + \dots + \lambda_{n-1} y_{n-1} + x + y_n) \\
&= V_n(y_1, \dots, y_n) V_n(y_1, \dots, y_{n-1}, x + y_n) \\
&= V_n(y_1, \dots, y_n) \left[\sum_{s=0}^{n-1} Q_{n-1,s}(y_1, \dots, y_{n-1})(x + y_n)^{2^s} \right]
\end{aligned}$$

(see Mùi [3, Appendix])

$$\begin{aligned}
&= V_n(y_1, \dots, y_n) \left[V_n(y_1, \dots, y_n) + \sum_{s=0}^{n-1} Q_{n-1,s}(y_1, \dots, y_{n-1}) x^{2^s} \right] \\
&= V_n^2 + \sum_{s=0}^{n-1} x^{2^s} \cdot V_n \cdot Q_{n-1,s}.
\end{aligned}$$

From this, Theorem A follows.

4. Proof of Theorem B

4.1. **Lemma.** *There exists uniquely an expansion*

$$\begin{aligned}
&V_{m+n+1}(x_1, \dots, x_m, y_1, \dots, y_n, z) \\
&\qquad = \sum_{s=0}^{n} P_s(x_1, \dots, x_m, y_1, \dots, y_n) V_{m+1}^{2^s}(x_1, \dots, x_m, z)
\end{aligned}$$

with $P_s \in \mathbf{Z}/2[x_1, \dots, x_m]^{GL_m} \otimes \mathbf{Z}/2[y_1, \dots, y_n]^{GL_n}$. *Furthermore, if*

$$P_s = \sum_{R=(r_1, \dots, r_m)} Q_{m,0}^{2^n - 2^s - (r_1 + \dots + r_m)} Q_{m,1}^{r_1} \cdots Q_{m,m-1}^{r_{m-1}} \cdot \psi_s^R(Q_{n,0}, \dots, Q_{n,n-1})$$

with $Q_{m,r} = Q_{m,r}(x_1, \dots, x_m)$, $Q_{n,t} = Q_{n,t}(y_1, \dots, y_n)$, *then*

$$St^R(Q_{n,s}) = \psi_s^R(Q_{n,0}, \dots, Q_{n,n-1}).$$

Proof. To prove the uniqueness of such an expansion, we suppose

$$\sum_{s=0}^{n} F_s(x_1, \dots, x_m, y_1, \dots, y_n) V_{m+1}^{2^s}(x_1, \dots, x_m, z) = 0$$

with the F_s polynomials of the variables indicated.

We remark that $\deg_z V_{m+1}^{2^s}(x_1, \dots, x_m, z) = 2^{m+s}$. Observe the coefficient of the leading term $z^{2^{m+n}}$ on the left-hand side of the above equation and we have $F_n = 0$. So, we get

$$\sum_{s=0}^{n-1} F_s V_{m+1}^{2^s} = 0.$$

Next, consider the leading term $z^{2^{m+n-1}}$ in the last equation, we obtain $F_{n-1} = 0$. Continuing this argument, we get $F_n = F_{n-1} = \cdots = F_0 = 0$.

Now we prove the existence of such an expansion. Recall that

$$V_{n+1}(y_1, \ldots, y_n, z) = \sum_{s=0}^{n} Q_{n,s}(y_1, \ldots, y_n) \cdot z^{2^s},$$

(see Mùi [3, Appendix]). With Proposition 2.2 in hand, apply $d_m^* P_m$ to the two sides of this equality and obtain

$$V_{m+n+1}(x_1, \ldots, x_m, y_1, \ldots, y_n, z) = \sum_{s=0}^{n} d_m^* P_m(Q_{n,s}) \cdot V_{m+1}^{2^s}(x_1, \ldots, x_m, z).$$

This is the desired expansion with

$$P_s = d_m^* P_m(Q_{n,s}) \in \mathbf{Z}/2[x_1, \ldots, x_m]^{GL_m} \otimes \mathbf{Z}/2[y_1, \ldots, y_n]^{GL_n}.$$

Moreover, according to Theorem 2.1, we have

$$d_m^* P_m Q_{n,s} = \sum_{R=(r_1, \ldots, r_m)} Q_{m,0}^{2^n - 2^s - (r_1 + \cdots + r_m)} Q_{m,1}^{r_1} \cdots Q_{m,m-1}^{r_{m-1}} \cdot St^R(Q_{n,s}).$$

So we get the last conclusion of the lemma

$$St^R(Q_{n,s}) = \psi_s^R(Q_{n,0}, \ldots, Q_{n,n-1}).$$

4.2. *Proof of Theorem* B. Apply Lemma 4.1 with $m = 1$ and remember that $St^{(r)} = Sq^r$. By the same argument as given in the proof of Theorem A, we have

$$\begin{aligned}
V_{n+2}(x, y_1, \ldots, y_n, z) &= V_{n+1}(y_1, \ldots, y_n, z) V_{n+1}(y_1, \ldots, y_n, x+z) \\
&= V_{n+1}^2(y_1, \ldots, y_n, z) \\
&\quad + V_{n+1}(y_1, \ldots, y_n, z) V_{n+1}(y_1, \ldots, y_n, x) \\
&= \sum_{s=0}^{n} Q_{n,s}^2 z^{2^{s+1}} + \sum_{s=0}^{n} Q_{n,s} z^s \cdot \sum_{t=0}^{n} Q_{n,t} x^t \\
&= \sum_{s=0}^{n} Q_{n,s}^2 (x^{2^s} z^{2^s} + z^{2^{s+1}}) + \sum_{r=0}^{n-1} Q_{n,r} (x^{2^n} z^{2^r} + x^{2^r} z^{2^n}) \\
&\quad + \sum_{0 \le r < t < n} Q_{n,r} Q_{n,t} (x^{2^r} z^{2^t} + x^{2^t} z^{2^r}).
\end{aligned}$$

Note that $V_2(x, z) = xz + z^2$. A simple computation leads us to

$$\begin{aligned} x^{2^n} z^{2^r} + x^{2^r} z^{2^n} &= x^{2^n-2^r}(x^{2^r} z^{2^r} + z^{2^{r+1}}) + x^{2^n-2^r-2^{r+1}}(x^{2^{r+1}} z^{2^{r+1}} + z^{2^{r+2}}) + \cdots \\ &\quad + x^{2^n-2^r-\cdots-2^{n-1}}(x^{2^{n-1}} z^{2^{n-1}} + z^{2^n}) \\ &= \sum_{s=r}^{n-1} x^{2^n-(2^{s+1}-2^r)} V_2^{2^s}(x, z), \\ x^{2^t} z^{2^r} + x^{2^r} z^{2^t} &= x^{2^t-2^r}(x^{2^r} z^{2^r} + z^{2^{r+1}}) + \cdots \\ &\quad + x^{2^t-2^r-\cdots-2^{t-1}}(x^{2^{t-1}} z^{2^{t-1}} + z^{2^t}) \\ &= \sum_{s=r}^{t-1} x^{2^t-(2^{s+1}-2^r)} V_2^{2^s}(x, z). \end{aligned}$$

So we get

$$\begin{aligned} V_{n+2}(x, y_1, \ldots, y_n, z) = \sum_{s=0}^{n} Q_{n,s}^2 V_2^{2^s} + \sum_{r \le s < n} x^{2^n-2^{s+1}+2^r} Q_{n,r} V_2^{2^s} \\ + \sum_{r \le s < t < n} x^{2^t-2^{s+1}+2^r} Q_{n,r} Q_{n,t} V_2^{2^s}. \end{aligned}$$

From Lemma 4.1, this implies

$$d^* P Q_{n,s} = Q_{n,s}^2 + \sum_{r \le s} x^{2^n-2^{s+1}+2^r} Q_{n,r} + \sum_{r \le s < t} x^{2^t-2^{s+1}+2^r} Q_{n,r} Q_{n,t}.$$

Theorem B follows.

Acknowledgment

I would like to thank the Mathematical Sciences Research Institute, Berkeley, California, for the hospitality and the ideal working conditions while I was visiting there in the academic year 1989–1990.

Note added in proof. Recently, H. E. A. Campbell informed me that T. Bisson independently showed Theorem B in a private letter sent to Campbell.

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