PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 114, Number 1, January 1992

GENERIC HEAT DIFFUSION IS SCALAR CONTROLLABLE

LAWRENCE MARKUS

(Communicated by Kenneth R. Meyer)

ABSTRACT. The partial differential equation for heat diffusion on a closed manifold M is approximately controllable by a single distributed controller, under generic conditions. But we also give examples, where M is a torus surface, for which no finite number of scalar controllers suffice.

Let M be a smooth closed n-manifold (C^∞ with $n \geq 2$, compact without boundary). Consider the set $\mathscr{G}$ of all Riemannian metric tensor fields of class C^k on M (for any fixed k on $n+4 \leq k \leq \infty$), so $\mathscr{G}$ with the C^k-topology has the topology of a complete metric space, and hence $\mathscr{G}$ is a Baire space; and let $\mathscr{F} = L^2(M, \mathbf{R})$, also topologized as a Baire space. For each pair $(g, f) \in \mathscr{G} \times \mathscr{F}$, consider the heat-diffusion control system on M, for the state $w(\cdot, t) \in L^2(M, \mathbf{R})$ at each time $t \geq 0$,

$$(*) \qquad \frac{\partial w}{\partial t} = \Delta_g w + f \cdot u(t).$$

Here Δ_g is the Laplace-Beltrami operator for the metric tensor g, and we choose $u(t) \in L^1_{\text{loc}}(0, \infty; \mathbf{R})$ as the scalar control function. We are interested in the approximate controllability of $(*)$; see [3].

We assert that the set

$$C = \{(g, f) \in \mathscr{G} \times \mathscr{F} \mid (*) \text{ is approximately controllable in } L^2(M, \mathbf{R}), \text{ and in arbitrarily short durations}\}$$

is residual (and hence dense) in $\mathscr{G} \times \mathscr{F}$. We paraphrase this assertion in the

Theorem. *Generic heat diffusion is scalar controllable.*

Proof. Define $\mathscr{G}_s = \{g \in \mathscr{G} \mid \Delta_g$ has only simple eigenvalues$\}$, and note that $\mathscr{G}_s$ is residual in $\mathscr{G}$; see [4]. Now consider the set $Z = \{(g, f) \in \mathscr{G} \times \mathscr{F} \mid g \in \mathscr{G}_s$ and f has all nonzero Fourier coefficients for the eigenbasis of $\Delta_g\}$.

Suppose that Z is of first category in the Baire space $\mathscr{G} \times \mathscr{F}$. In this case [2, §22V], there exists some $g_0 \in \mathscr{G}_s$ such that $Z \cap \mathscr{F}^{g_0}$ is of first category in the "vertical section" $\mathscr{F}^{g_0} = (g_0) \times \mathscr{F} \subset \mathscr{G} \times \mathscr{F}$. This is a contradiction since the set $\{f \in \mathscr{F} \mid f$ has all nonzero Fourier coefficients for a prescribed orthonormal

Received by the editors June 12, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 58G11; Secondary 93C20.
Research done under NSF Grant DMS 87-22402.

basis in $L^2(M, \mathbf{R})\}$ is clearly residual in $\mathcal{F}$. Hence Z must be residual. But $C \supset Z$, see [1, 3], so C is also residual in $\mathcal{G} \times \mathcal{F}$. $\square$

Similar results hold for the case where M is a region in some ambient manifold, where $\overline{M}$ is compact and Dirichlet conditions apply on the smooth boundary ∂M—and, for more general linear parabolic systems.

Example. Consider the heat-diffusion distributed parameter control system

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + f_1(x, y)u_1(t) + \cdots + f_m(x, y)u_m(t) \quad \text{for } t \geq 0$$

on a locally flat torus surface T^2 defined by the translations of the Euclidean (x, y)-plane

$$\begin{matrix} x \to x + 2\pi\alpha, & x \to x, \\ y \to y, & y \to y + 2\pi\beta, \end{matrix} \quad \text{fixed real } \alpha > 0, \beta > 0.$$

Here the m functions $f_j(x, y) \in L^2(T^2)$ and the scalar controllers $u_j(t) \in L^1_{\text{loc}}(0, \infty; \mathbf{R})$ (for $j = 1, \dots, m$) are as before. A necessary condition for the approximate controllability of this system is that each eigenspace E_λ of the Laplacian has a dimension $\leq m$; see [3]. We assume that $\alpha/\beta = p/q$ is a rational number (here p, q are positive integers), and show that $\sup \dim E_\lambda = \infty$.

Take positive integers a_1, b_1 so that the number $(a_1)^2 + (b_1)^2$ can be written as the sum of two integral squares in more than m distinct ways. Define $a = a_1 p$, $b = b_1 q$, and compute

$$\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} = \frac{p^2}{\alpha^2}[(a_1)^2 + (b_1)^2].$$

But $\phi(x, y) = \sin(ax/\alpha)\sin(by/\beta)$ is an eigenfunction for the Laplacian, with eigenvalue $\lambda_{ab} = -[a^2/\alpha^2 + b^2/\beta^2] = -(p^2/\alpha^2)[a_1^2 + b_1^2]$. Thus, for each pair of integers $\hat{a}, \hat{b}$ with $\hat{a}^2 + \hat{b}^2 = a_1^2 + b_1^2$, we have a corresponding pair $\hat{a}p, \hat{b}q$ defining an eigenfunction $\hat{\phi}(x, y) = \sin(\hat{a}px/\alpha)\sin(\hat{b}qy/\beta)$ with the same eigenvalue $\lambda_{ab} = -(p^2/\alpha^2)[\hat{a}^2 + \hat{b}^2]$. Therefore $\dim E_{\lambda_{ab}} > m$, and clearly $\sup \dim E_\lambda = \infty$.

On each smooth closed n-manifold M define the subset of Riemann metrics

$$\mathcal{G}_r = \{g \in \mathcal{G} \mid \text{eigenspaces } E_\lambda \text{ of } \Delta_g \text{ satisfy } \sup \dim E_\lambda = \infty\}.$$

It would be interesting to know under what circumstances $\mathcal{G}_r$ is dense in $\mathcal{G}$.

References

1. H. O. Fattorini, *On complete controllability of liner systems*, J. Differential Equations **3** (1967), 391–402.
2. K. Kuratowski, *Topology*, vol. 1, Academic Press, New York, 1966.
3. L. Markus, *Introduction to the theory of distributed control systems*, Distributed Parameter Control Systems: Trends and Application Systems, Marcel Dekker, NY, 1990.
4. K. Uhlenbeck, *Generic properties of eigenfunctions*, Amer. J. Math. **98** (1976), 1059–1078.

Department of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455