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# A NOTE ON THE $L^p$ ANALOGUE OF THE "ZERO-TWO" LAW

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ABSTRACT. It was proved by R. Wittmann [2] that, given a positive linear contraction of $L^p$ $(1 \le p < \infty)$, $\sup_{\|f\|_p \le 1} \lim_{n\to\infty} \|T^n f - T^{n+1} f\|$ is either $\ge \alpha_p$ or 0; the (best possible) value of $\alpha_p$ is the $l_p$-norm of a certain $3 \times 3$ matrix. In this paper $\alpha_p$ is explicitly expressed as a function of $p$.

Let $T$ be a positive linear contraction of $L^1$. Ornstein and Sucheston [1] proved that $\sup_{\|f\|_1 \le 1} \lim_{n\to\infty} \|T^n f - T^{n+1} f\|_1$ is either 0 or 2. Wittmann [2] showed that, if $T$ is a positive linear contraction of $L^p$ $(1 \le p < \infty)$, then the analogous expression is either $\ge \alpha_p$ or 0. Here the constants $\alpha_p$ are defined by

$$\alpha_p = \sup_{0 \le x \le y} \left( \frac{(1+x)^p + (1+y)^p + (y-x)^p}{1 + x^p + y^p} \right)^{1/p}, \qquad 1 \le p < \infty,$$

and cannot be replaced by any smaller numbers. Equivalently [2, Ex. 3.1], $\alpha_p$ is the operator norm of the linear transformation $S$ of the space $\mathbb{R}^3$, endowed with $l_p$-norm, given by

$$S(x, y, z) = (x-y, y-z, z-x), \qquad (x, y, z) \in \mathbb{R}^3.$$

A lot of information was obtained regarding the $\alpha_p$'s, including [2, p. 21]:

(a) $\alpha_p \ge (1 + 2^{p-1})^{1/p}$,
(b) $\alpha_p = \alpha_q$, $(1 < p, q < \infty,\ 1/p + 1/q = 1)$,
(c) $\alpha_2 = \sqrt{3}$.

In this paper we determine the exact value of $\alpha_p$; namely, we prove the following:

**Theorem.** *The constants* $\alpha_p$ *are given by*

$$\alpha_p = \begin{cases} (1 + 2^{p-1})^{1/p}, & 1 \le p \le 2, \\ (1 + 2^{1/(p-1)})^{1-1/p}, & 2 \le p < \infty. \end{cases} \tag{1}$$

*Proof.* In view of the second definition of $\alpha_p$ we have

$$\alpha_p = \max\{|x-y|^p + |y-z|^p + |z-x|^p : |x|^p + |y|^p + |z|^p = 1\}.$$

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Denote by $S_1$ the unit $l_p$-sphere in $\mathbb{R}^3$. Define $F: S_1 \to \mathbb{R}$ by

$$F(x, y, z) = |x-y|^p + |y-z|^p + |z-x|^p, \qquad (x, y, z) \in S_1.$$

We are looking for the maximum of $F$ on $S_1$. Inverting the signs of all variables if necessary, we may restrict our attention to points $(x, y, z)$ with at least two nonnegative coordinates. Since $F$ is symmetric, we may also assume $y \geq x \geq z$. If $z > 0$ then clearly $F(x, y, z) < F(x, y, -z)$. It suffices therefore to find the maximum of $F$ on the set

$$Q = \{(x, y, z) \in S_1 : y \geq x \geq 0 \geq z\}.$$

Note that on $Q$ we have $x^p + y^p + (-z)^p = 1$ and

$$F(x, y, z) = (y-x)^p + (y-z)^p + (x-z)^p.$$

Let us look first for local maxima of $F$ in the interior $\mathring{Q}$ of $Q$. Using Lagrange multipliers we see that if $F$ has a local maximum at $(x, y, z) \in \mathring{Q}$ then

$$-(y-x)^{p-1} + (x-z)^{p-1} = \lambda x^{p-1}, \tag{2}$$

$$(y-x)^{p-1} + (y-z)^{p-1} = \lambda y^{p-1}, \tag{3}$$

$$-(y-z)^{p-1} - (x-z)^{p-1} = -\lambda(-z)^{p-1}, \tag{4}$$

for a certain $\lambda$. Adding up all three equations by sides we obtain

$$\lambda(x^{p-1} + y^{p-1} - (-z)^{p-1}) = 0.$$

From (3) it is clear that $\lambda \neq 0$, and hence the last equation yields

$$x^{p-1} + y^{p-1} = (-z)^{p-1}. \tag{5}$$

Now (2) and (3) imply

$$x^{p-1}[(y-x)^{p-1} + (y-z)^{p-1}] = y^{p-1}[-(y-x)^{p-1} + (x-z)^{p-1}].$$

Rearranging and applying (5) we get

$$x^{p-1}(y-z)^{p-1} + (-z)^{p-1}(y-x)^{p-1} = y^{p-1}(x-z)^{p-1}.$$

Setting $u = x(y-z)$ and $v = -z(y-x)$, we can write this as

$$u^{p-1} + v^{p-1} = (u+v)^{p-1}.$$

However, as both $u$ and $v$ are strictly positive on $\mathring{Q}$, the last equation has no solution unless $p = 2$. Hence for $p \neq 2$ the maximum is attained on the boundary $\partial Q$ of $Q$.

Obviously, $\partial Q = B_1 \cup B_2 \cup B_3$, where $B_1 = Q \cap \{x = 0\}$, $B_2 = Q \cap \{x = y\}$, and $B_3 = Q \cap \{z = 0\}$. The restriction of $F$ to each of the $B_i$'s can be readily expressed as a function of a single variable and its maximum is easily found. Thus:

$$\max_{(x,y,z)\in B_1} F(x, y, z) = F(0, 2^{-1/p}, -2^{-1/p}) = 1 + 2^{p-1}.$$

$$\begin{aligned}\max_{(x,y,z)\in B_2} F(x, y, z) &= F(1/(2 + 2^{p/(p-1)})^{1/p}, 1/(2 + 2^{p/(p-1)})^{1/p}, \\ &\qquad - 1/(1 + 2^{-1/(p-1)})^{1/p}) \\ &= (1 + 2^{1/(p-1)})^{p-1}.\end{aligned}$$

$$\max_{(x,y,z)\in B_3} F(x, y, z) = F(0, 1, 0) = 2.$$

Except for $p = 1$ (where the maximum on $B_2$ should be interpreted as 2), the third of these maxima is clearly inferior to the former two. Hence to conclude the proof of the theorem it suffices to show that

$$(6)\quad \max\{1 + 2^{p-1}, (1 + 2^{1/(p-1)})^{p-1}\} = \begin{cases} 1 + 2^{p-1}, & 1 < p \le 2, \\ (1 + 2^{1/(p-1)})^{p-1}, & 2 \le p < \infty. \end{cases}$$

In fact, suppose $p > 2$. Write $t = p - 1$. For appropriately chosen $\alpha > 2$ and $0 < \beta < 1$

$$\begin{aligned}
&(1 + 2^{1/(p-1)})^{p-1} - (1 + 2^{p-1}) \\
&\quad = (1 + 2^{1/t})^t - 2^t - 1 = (1 + 2^{1/t} - 2) \cdot t\alpha^{t-1} - 1 \\
&\quad > t2^{t-1}(2^{1/t} - 1) - 1 = t2^{t-1}((1 + 1)^{1/t} - 1) - 1 \\
&\quad = t2^{t-1}\left(1 + \binom{1/t}{1} + \binom{1/t}{2} + \binom{1/t}{3}(1 + \beta)^{1/t} - 1\right) - 1 \\
&\quad \ge 2^{t-1}(1/2 + 1/2t) - 1 = (2^{t-1}(1 + t) - 2t)/2t \\
&\quad \ge (2^{t-1} \cdot 2\sqrt{t} - 2t)/2t = (2^{t-1} - \sqrt{t})/\sqrt{t} \\
&\quad \ge (4^{t-1} - t)/(2^{t-1} + \sqrt{t})t \ge (e^{t-1} - t)/(2^t + t)t > 0.
\end{aligned}$$

For $1 < p < 2$ define $q$ by $1/p + 1/q = 1$. Since $q > 2$ the last inequality implies

$$1 + 2^{p-1} = 1 + 2^{1/(q-1)} > (1 + 2^{q-1})^{1/(q-1)} = (1 + 2^{1/(p-1)})^{p-1},$$

which completes the proof of (6). This concludes the proof of the theorem for $p \ne 2$, and by continuity for $p = 2$ as well.

*Remark.* As seen from the theorem, $\alpha_p$ is not differentiable as a function of $p$ at the point 2. The reason for this is that, whereas for $p < 2$ the maximum of $F$ is attained on $B_1$, for $p > 2$ it is attained on $B_2$. For $p = 2$

$$(x - y)^2 + (y - z)^2 + (z - x)^2 = 2 - 2xy - 2xz - 2yz = 3 - (x + y + z)^2,$$

so the maximum is attained along the whole circle formed by the intersection of the unit sphere $\mathbf{S}^2$ and the plane $x + y + z = 0$.

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## References

1. D. Ornstein and L. Sucheston, *An operator theorem on $L^1$ convergence to zero with applications to Markov kernels*, Ann. Math. Statist. **41** (1970), 1631–1639.
2. R. Wittmann, *Analogues of the "zero-two" law for positive linear contractions in $L^p$ and $C(X)$*, Israel J. Math. **59** (1987), 8–28.

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