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CHAIN CONDITIONS ON ESSENTIAL SUBMODULES

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ABSTRACT. For $\aleph$ an infinite cardinal and M a unital right module over a ring R with 1 or an object in an $\mathscr{AB}5$ category, we show that every well ordered ascending (respectively descending) chain of essential submodules of M has cardinality less than $\aleph$ if and only if every well ordered ascending (respectively descending) chain of submodules of $M/\text{socle}(M)$ has cardinality less than $\aleph$. We use this to show that a CS module with an $\aleph$-chain condition on essential submodules is a direct sum of a module with that same chain condition on all submodules plus a semisimple module. Thus a CS module with fewer than $\aleph$ generators has an $\aleph$-chain condition on essential submodules if and only if it has that same $\aleph$-chain condition on all submodules. As an application in the case of an $\aleph_0$-chain condition, we describe the endomorphism ring of a continuous module with acc on essential submodules.

Chain conditions on a module (or object in an $\mathscr{AB}5$ category) appear in many contexts. We say that a partially ordered set $\mathscr{X}$ has the ascending (respectively descending) $\aleph$-chain condition iff for every ordinal κ such that there is a chain $\{ N_\alpha \mid \alpha < \kappa \}$ of subsets of $\mathscr{X}$ with $N_\beta < N_\alpha$ (respectively $N_\beta > N_\alpha$) for all $\beta < \alpha \in \kappa$, we have $|\kappa| < \aleph$. Thus the usual acc and dcc are $\aleph_0$-chain conditions. Bass has shown ([3]) that any commutative Noetherian ring has the descending $\aleph_1$-chain condition on ideals. In contrast, Jategaonkar has shown ([7]) that right principal ideal domains can have descending chains of two sided ideals of any preassigned cardinality. Lawrence has shown ([8]) that a right self injective ring R of cardinality $\aleph$ has the ascending $\aleph$-chain condition on annihilator right ideals. The Hopkins-Levitski theorem says that a ring with the descending $\aleph_0$-chain condition on right ideals has the ascending $\aleph_0$-chain condition on right ideals. There are valuation domain examples where the descending $\aleph_1$-chain condition does not imply the ascending $\aleph_1$-chain condition. Several authors have looked at the usual ($\aleph_0$-) chain conditions on the subset of all essential submodules of a module and have obtained or used an $\aleph_0$ version of some of our theorems in the special case of ordinary acc and/or dcc. We mention, for example [1, 4, 5, 6]. In this note we show that, for any infinite cardinal $\aleph$, an $\aleph$-chain condition on essential submodules of a module M is very close to that $\aleph$-chain condition on all submodules. Specifically, M has an

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$\aleph$-chain condition on all submodules if and only if M has that $\aleph$-chain condition on essential submodules and on sets of independent submodules. Also, M has an $\aleph$-chain condition on essential submodules if and only if $M/\text{socle}(M)$ has that $\aleph$-chain condition on all submodules. We apply the results to CS modules, that is, modules for which any submodule is essential in a direct summand. We show that any such module with an $\aleph$-chain condition on essentials is a direct sum of a module with the $\aleph$-chain condition on all submodules and a semisimple module. In the case of the ascending $\aleph_0$-chain condition on essential submodules of a continuous module M this enables us to describe the endomorphism ring of M.

M will denote a fixed R-module or object in an $\mathcal{AB}5$ category, and $\aleph$ an infinite cardinal. For any set $\mathcal{X}$, $|\mathcal{X}|$ will denote the cardinality of $\mathcal{X}$. $N \subseteq M$ will mean N is a submodule of M. N is an essential submodule of M, that is N has nonzero intersection with each nonzero $L \subseteq M$, will be denoted $N \subseteq' M$. For $N \subseteq M$, a complement of N is a submodule of M maximal in $\{L \subseteq M | L \cap N = 0\}$. A module is CS if and only if every complement submodule is a direct summand. A family of submodules $\{N_i \mid i \in \mathcal{I}\}$ is called independent if the sum $\sum_{i \in \mathcal{I}} N_i$ is direct and the N_i are all nonzero. Proofs are phrased in terms of (right) modules, but they are valid in an arbitrary $\mathcal{AB}5$ category. For example, a categorical definition of $\aleph$-generated can be found in [10]. Reference [2] is a good source of background material. Several well known or easily established results are included as lemmas for convenience in referencing, sometimes with a brief sketch of a proof.

Lemma 1. *If M has an $\aleph$-chain condition on essential submodules, then so does every submodule and every quotient module of M.*

Sketch of proof. Let $K \subseteq M$, and let L be a complement of K in M. Then adding L to each member of a chain of submodules essential in K gives a chain of submodules essential in M, so submodules inherit chain conditions on essentials. The inverse image of an essential submodule of M/K is essential in M, so quotients also inherit chain conditions on essentials.

Theorem 1. *Let M be a module and $\aleph$ an infinite cardinal such that*

(1) *Any independent family of submodules of M has fewer than $\aleph$ members.*
(2) *The set of all essential submodules of M has the ascending* (*respectively descending*) *$\aleph$-chain condition.*

Then the set of all submodules of M has the ascending (*respectively descending*) *$\aleph$-chain condition. Conversely, an $\aleph$-chain condition on all submodules of M implies* (1) *and the same chain condition in* (2).

Proof. The proof will be divided into three parts. To avoid interrupting the flow of ideas with messy details, only the first will be given here. It handles the case when $\aleph$ is a regular cardinal. This portion of the proof is straightforward and gives insight into why one would expect the theorem to be true. The proofs for the ascending and descending $\aleph$-chain condition when $\aleph$ is not regular are deferred to an appendix.

Part 1. $\aleph$ *regular.* Let $\{N_\alpha | \alpha < \kappa\}$ be an ascending (respectively descending) chain of distinct submodules of M. Define an equivalence relation $\sim$ on κ by $\alpha \sim \beta$ iff $N_\alpha \subseteq' N_\beta$ or $N_\beta \subseteq' N_\alpha$. The set $\{N_\alpha\}$ corresponding to members of an equivalence class is an ascending (respectively descending) chain of submodules essential in the union of the set. Thus by Lemma 1, there are fewer than $\aleph$ members in each equivalence class. Also, each class has a smallest member $\overline{\alpha}$. The set $\mathscr{C}_{\sim} = \{\sim\text{-class representatives } \overline{\alpha}\}$ is a subchain of κ and so well ordered. Let $\overline{\alpha^+}$ denote the successor of $\overline{\alpha}$ in this subchain. For each $\overline{\alpha^+} \in \mathscr{C}_{\sim}$, the smaller of $\{ N_{\overline{\alpha}}, N_{\overline{\alpha^+}}\}$ has a nonzero complement $L_{\overline{\alpha}}$ in the larger. These $L_{\overline{\alpha}}$ are clearly independent and in one-to-one correspondence with the set of successors in $\mathscr{C}_{\sim}$. If $\mathscr{C}_{\sim}$ is infinite it has the same cardinality as the set of successors in it. Thus there are fewer than $\aleph$ equivalence classes by (1). If $\aleph$ is regular, then κ, being a union of less than $\aleph$ classes of less than $\aleph$ elements, must have $|\kappa| < \aleph$. $\square$

The converse of the theorem is clear.

Lemma 2. *A module N is semisimple $\iff$ N has no proper essential submodules.*

Sketch of proof. If N is semisimple, then every submodule is a direct summand of N and so not essential unless equal to N. Conversely, any $K \subseteq N$ has a complement $L \subseteq N$. Then $K \oplus L \subseteq' N$, so if N has no proper essential submodules, K is a direct summand of N.

Theorem 2. *Let M be a module with either the ascending or descending $\aleph$-chain condition on essential submodules. Then every direct sum $\oplus_{\alpha\in\kappa} E_\alpha \subseteq M$, where each E_α is not semisimple, has fewer than $\aleph$ E_α.*

Proof. Let $\oplus_{\alpha\in\kappa} E_\alpha$ be an infinite direct sum of nonsemisimple submodules of M. By adding a complement of $\oplus_{\alpha\in\kappa} E_\alpha$ to one of the E_α if necessary, we may assume that $\oplus_{\alpha\in\kappa} E_\alpha$ is essential in M. Without loss of generality we also may assume that κ is an ordinal.

For each $\alpha \in \kappa$, select a proper essential submodule of E_α and denote it by N_α. Set

$$\mathscr{U}_\alpha = \left(\bigoplus_{\beta<\alpha} N_\beta\right) \oplus \left(\bigoplus_{\beta\geq\alpha} E_\beta\right)$$

$$\mathscr{V}_\alpha = \left(\bigoplus_{\beta<\alpha} E_\beta\right) \oplus \left(\bigoplus_{\beta\geq\alpha} N_\beta\right).$$

Clearly the $\mathscr{U}_\alpha$ form a descending chain and the $\mathscr{V}_\alpha$ an ascending chain of order type κ. If $\mathscr{U}_\alpha = \mathscr{U}_{\alpha+1}$ (respectively $\mathscr{V}_\alpha = \mathscr{V}_{\alpha+1}$), taking complements modulo $\bigoplus_{\beta<\alpha} N_\beta \oplus \bigoplus_{\beta>\alpha} E_\beta$ (respectively $\bigoplus_{\beta<\alpha} E_\beta \oplus \bigoplus_{\beta>\alpha} N_\beta$) gives $N_\alpha = E_\alpha$ since $E_\alpha \supseteq N_\alpha$. But N_α was a proper submodule of E_α, so the $\mathscr{U}_\alpha$ (respectively $\mathscr{V}_\alpha$) are all distinct. Thus either the ascending or descending $\aleph$-chain condition (2) implies that $|\kappa| < \aleph$. $\square$

Lemma 3. *If $A \subseteq B \subseteq M$ and $L \subseteq M$, then $A = B \iff A \cap L = B \cap L$ and $A + L = B + L$. In particular, if κ is well ordered and $\{N_\alpha | \alpha < \kappa\}$ is an increasing or decreasing chain of submodules of M, then for any $L \subseteq M$, if $N_\alpha + L = N_\beta + L$ (respectively $N_\alpha \cap L = N_\beta \cap L$) for all β with $\alpha \leq \beta < \gamma$, then $\{N_\beta \cap L | \alpha \leq \beta < \gamma\}$ (respectively $\{N_\beta + L | \alpha \leq \beta < \gamma\}$) is an increasing or decreasing chain of submodules of L (respectively M/L).*

Theorem 3. *M has the ascending (respectively descending) $\aleph$-chain condition* *on essential submodules* $\Longleftrightarrow$ $M/\text{socle}(M)$ *has the ascending (respectively descending)* $\aleph$*-chain condition on all submodules.*

Proof. Let S denote the socle of M. Since any essential submodule of M must contain S, $\Longleftarrow$ follows from Lemma 3. For the converse, there is a submodule $K \subseteq M$ with $S \oplus K \subseteq' M$. Then $M/(K \oplus S)$ has the appropriate $\aleph$-chain condition. By Lemma 3, M/S has an $\aleph$-chain condition if and only if both $M/(S \oplus K)$ and $(S \oplus K)/S \cong K$ do. But K has zero socle, so by Theorem 2, every independent family of submodules of K has fewer than $\aleph$ elements. Thus Theorem 1 shows that K has the appropriate $\aleph$-chain condition. □

Lemma 4. *If* S *is the socle of a direct sum* $\bigoplus N_\alpha$, *then* $S = \bigoplus \text{socle}(N_\alpha)$. *Hence* $(\bigoplus N_\alpha)/S \cong \bigoplus (N_\alpha/\text{socle}(N_\alpha))$.

Sketch of proof. A homomorphic image of a semisimple is semisimple, so the socle of each N_α contains the projection of S to N_α.

Theorem 4. *Let* M *be a* CS *module. Then* M *has the ascending (respectively descending)* $\aleph$*-chain condition on essential submodules* $\Longleftrightarrow$ M *is a direct sum* $K \oplus L$ *where* K *has the ascending (respectively descending)* $\aleph$*-chain condition on all submodules and* L *is semisimple.*

Proof. $\Longleftarrow$ is clear since any essential submodule contains the socle.

For $\Longrightarrow$, by Theorem 3 $M/\text{socle}(M)$ has the appropriate $\aleph$-chain condition. Let $\{\overline{N}_\alpha \mid \alpha \in \kappa\}$ be a maximal independent family of submodules of $M/\text{socle}(M)$ where each $\overline{N}_\alpha$ is finitely generated. Then $|\kappa| < \aleph$ and there is a family $\{N_\alpha \subseteq M \mid \alpha \in \kappa\}$ of finitely generated N_α such that $N_\alpha + \text{socle}(M) = \overline{N}_\alpha$ for each $\alpha \in \kappa$. Let K be a maximal essential extension of $\sum_{\alpha\in\kappa} N_\alpha$ in M. Then K is a direct summand of M since M is CS. Let $M = K \oplus L$. By Lemma 4, $\text{socle}(M) = \text{socle}(K) \oplus \text{socle}(L)$. Since $K/\text{socle}(K) \subseteq' M/\text{socle}(M) = K/\text{socle}(K) \oplus L/\text{socle}(L)$, $L = \text{socle}(L)$ is semisimple. We next show that $\text{socle}(K)$ has the ascending $\aleph$-chain condition. Since $\text{socle}(K)$ is semisimple it has a well-defined dimension as a sum of simples, so we need only show that $\text{socle}(K)$ has no independent family with $\aleph$ members. Assume $\text{socle}(K)$ does contain an independent family with $\aleph$ elements. Since $\aleph \cdot \aleph = \aleph$, $\text{socle}(K)$ contains some independent family $\{T_\beta\}$ of cardinality $\aleph$, where each T_β cannot be generated by fewer than $\aleph$ elements. For each β, let U_β be a maximal essential extension of T_β in M. By Lemma 4, $\sum_\beta U_\beta/T_\beta$ is direct in $M/\text{socle}(M)$. Thus for some γ, $U_\gamma = T_\gamma$, so T_γ is a direct summand of M and hence of any submodule containing it. $\sum_{\alpha\in\kappa} N_\alpha$ is a submodule of M generated by fewer than $\aleph$ elements and it contains T_γ, so T_γ is generated by fewer than $\aleph$ elements, a contradiction. Thus $\text{socle}(K)$ must be generated by fewer than $\aleph$ elements, so both $K/\text{socle}(K)$ and $\text{socle}(K)$ have the appropriate $\aleph$-chain condition and, by Lemma 3, so does K. □

We remark that [4] has a weaker version of this result for $\aleph = \aleph_0$.

Corollary 5. *Let* M *be* CS *and* $\aleph'$*-generated for some* $\aleph' < \aleph$. *Then* M *has an* $\aleph$*-chain condition on essential submodules if and only if* M *has that* $\aleph$*-chain condition on all submodules.*

Proof. The semisimple L in Theorem 4 must be $\aleph'$-generated. It therefore has dimension $\leq \aleph' < \aleph$ and so both $\aleph$-chain conditions hold for L. Apply Lemma 3. □

Corollary 6. *If the ring* R *has every complement right ideal a direct summand and an* $\aleph$*-chain condition on essential right ideals, then* R *has that* $\aleph$*-chain condition on all right ideals.*

Proof. 1 is less than any infinite cardinal. □

Corollary 6 plus the Hopkins-Levitzki theorem show that a right CS ring with the descending $\aleph_0$-chain condition on essential right ideals has finite composition length on the right.

Lemma 5. *Let M be a module that has acc on essential submodules. Then*

$$\mathbf{J} = \{ \lambda \in \mathrm{End}(M) \mid \mathrm{kernel}(\lambda) \subseteq' M \}$$

is a left T-nilpotent ideal of $\mathrm{End}(M)$.

Sketch of proof. It is standard that $\mathbf{J}$ is a two-sided ideal of $\mathrm{End}(M)$. Let $\{ \lambda_i \mid i \in \omega \} \subseteq \mathbf{J}$. By acc, there is an $n \in \omega$ with

$$\mathrm{kernel}(\lambda_{n+1}\lambda_n \cdots \lambda_0) = \mathrm{kernel}(\lambda_n \cdots \lambda_0).$$

Then $\mathrm{kernel}(\lambda_{n+1}) \bigcap \mathrm{image}(\lambda_n \cdots \lambda_0) = 0$, so by the definition of $\mathbf{J}$, $\lambda_n \cdots \lambda_0 = 0$. □

Corollary 7. *If M is a continuous module with acc on essential submodules, then* $\mathrm{End}(M)$ (*written on the left*) *is of the form*

$$\mathrm{End}(M) = \begin{pmatrix} \mathscr{S} & 0 \\ B & \Lambda \end{pmatrix}$$

where $\mathscr{S}$ is semiprimary, Λ is a direct product of full rings of linear transformations and B is a Λ-$\mathscr{S}$ bimodule.

Proof. By Corollary 4, $M = K \oplus L$ where K is Noetherian and L is semisimple. By taking a maximal semisimple direct summand of the Noetherian module K and adding it to L we may assume that K has no simple direct summand. Then K is Noetherian and continuous, so $\mathscr{S} = \mathrm{End}(K)$ is semiperfect and $\mathbf{J} = \{\lambda : K \longrightarrow K \mid \mathrm{kernel}(\phi) \subseteq' K\}$ is its radical. But acc on submodules of M implies there is an $n \in \omega$ such that $(0 : \mathbf{J}^n) = (0 : \mathbf{J}^k)$ for all $k > n$ where $(0 : N)$ denotes the annihilator of N in M. Assume we have a sequence $\{ \lambda_0, \lambda_1, \ldots, \lambda_k \}$ of elements of $\mathbf{J}$ such that $\mathbf{J}^n\lambda_k \cdots \lambda_0 M \neq 0$. Then $\mathbf{J}^{n+1}\lambda_k \cdots \lambda_0 M \neq 0$ so there is a $\lambda_{k+1} \in \mathbf{J}$ with $\mathbf{J}^n\lambda_{k+1} \cdots \lambda_0 M \neq 0$. By T-nilpotence, this cannot happen so $\mathbf{J}^n = 0$ (see [2, p. 324]) and $\mathscr{S}$ is semiprimary. L is semisimple, so $\Lambda = \mathrm{End}(L)$ is a product of the endomorphism rings of the homogeneous components of L, that is, rings of the form

$$\mathrm{Hom}\left(\bigoplus_{i \in \mathscr{I}} S_i, \bigoplus_{j \in \mathscr{I}} S_j \right) \cong \prod_i \bigoplus_j \mathrm{Hom}(S_i, S_j)$$

which is just column finite matrices over the division ring $\mathrm{Hom}(S, S)$. B is the Λ-$\mathscr{S}$ bimodule $\mathrm{Hom}(K, L)$. M is continuous and K has no simple direct summands, so there can be no maps from L to K. □

If M in Corollary 7 is the ring itself, we get the theorem in [6] that a continuous ring with acc on essential submodules has dcc. However, that is a very special case. There is an example of an injective module M with acc but not dcc (see [9]). Its endomorphism ring happens to be a division ring which is most certainly semiprimary.

Appendix. Proof of Theorem 1 for nonregular cardinals $\aleph$

This appendix contains the second (for ascending chains) and third (for descending chains) parts of the proof of Theorem 1. We now assume that $\aleph$ is not regular.

Theorem 1. *Let M be a module and $\aleph$ an infinite cardinal such that*

(1) *Any independent family of submodules of M has fewer than $\aleph$ members.*
(2) *The set of all essential submodules of M has the ascending* (*respectively descending*) *$\aleph$-chain condition.*

Then the set of all submodules of M has the ascending (*respectively descending*) *$\aleph$-chain condition. Conversely, an $\aleph$-chain condition on all submodules of M implies* (1) *and the same chain condition in* (2).

Continuation of the proof. As in Part 1 for the regular case, we let $\{N_\alpha | \alpha < \kappa\}$ be an ascending (respectively descending) chain of distinct submodules of M. For all α, $\beta < \kappa$, $\alpha \sim \beta$ iff $N_\alpha \subseteq' N_\beta$ or $N_\beta \subseteq' N_\alpha$. $\overline{\alpha}$ denotes the smallest member in its equivalence class and $\overline{\alpha^+}$ the successor of $\overline{\alpha}$ in the set $\mathscr{C}_\sim = \{\sim\text{-class representatives } \overline{\alpha}\}$.

Part 2. *Ascending chains.* Now assume the ascending $\aleph$-chain condition on essentials in (2), where $\aleph$ need not be regular. Since adjoining a new N_λ for each limit ordinal $\lambda < \kappa$ does not increase cardinality, we may assume that the chain $\{ N_\alpha \mid \alpha < \kappa \}$ is smooth, that is, for a limit ordinal λ, N_λ is the union of the preceding N_β. As in Part 1, let $L_{\overline{\alpha}}$ be a complement of $N_{\overline{\alpha}}$ in $N_{\overline{\alpha^+}}$. Let K be a complement of $\bigcup_{\alpha<\kappa} N_\alpha$ in M, and set

$$L = \left(\sum_{\mathscr{C}_\sim} L_{\overline{\alpha}} \right) \oplus K.$$

We claim that L is essential in M. Indeed, let Z be a nonzero submodule of M. Then $Z \cap (K \oplus \bigcup_{\alpha<\kappa} N_\alpha) \neq 0$, so unless $\kappa = 0$, using smoothness we get $Z \cap (K \oplus N_\alpha) \neq 0$ for some $\alpha < \kappa$ in the $\sim$-class of the $\mathscr{C}_\sim$-successor $\overline{\alpha_0{}^+}$. Then by definition of $L_{\overline{\alpha_0}}$ we get

$$Z \cap (K \oplus L_{\overline{\alpha_0}} \oplus N_{\overline{\alpha_0}}) \neq 0.$$

Again using smoothness at limit ordinals, if $\overline{\alpha_0} \neq 0$ there is an $\alpha < \overline{\alpha_0{}^+}$ with α in the $\sim$-class of a $\mathscr{C}_\sim$-successor $\overline{\alpha_1{}^+} < \overline{\alpha_0{}^+}$ such that

$$Z \cap (K \oplus L_{\overline{\alpha_0}} \oplus L_{\overline{\alpha_1}} \oplus N_{\overline{\alpha_1}}) \neq 0.$$

Continuing in this manner we get a descending sequence $\{\overline{\alpha_i}\} \subset \kappa$ that must terminate since κ is well ordered. If it terminates at $\overline{\alpha_n}$, then

$$Z \cap \left(K \oplus \sum_{i=0}^{n} L_{\overline{\alpha_i}} \right) \neq 0.$$

This shows that $L \subseteq' M$. We thus get a well ordered nondecreasing chain

$$\{ \overline{N_\alpha} = L + N_\alpha \big| \alpha < \kappa \}$$

of essential submodules of M. We observe that, for each $\alpha \sim \overline{\alpha}$, $N_\alpha \supseteq \sum_{\overline{\beta}<\overline{\alpha}} L_{\overline{\beta}}$. Also, either $N_\alpha \subset' N_{\alpha+1}$ or $\alpha+1=\overline{\alpha^+}$ and $(N_\alpha \oplus L_{\overline{\alpha}}) \subseteq' N_{\alpha+1}$. If $N_\alpha \subset' N_{\alpha+1}$ then $\overline{N_\alpha} \neq \overline{N_{\alpha+1}}$ since $K \oplus \sum_{\overline{\gamma}\geq\overline{\alpha}} L_{\overline{\gamma}}$ is a direct sum complement of $N_{\alpha+1}$ and N_α in $\overline{N_{\alpha+1}}$ and $\overline{N_\alpha}$ respectively, and $N_{\alpha+1} \supset N_\alpha$. If $(N_\alpha \oplus L_{\overline{\alpha}}) \subseteq' N_{\alpha+1}$ then, by the same reasoning, $\overline{N_{\alpha+1}} = \overline{N_\alpha}$ if and only if $N_{\alpha+1} = (N_\alpha \oplus L_{\overline{\alpha}})$.

Thus there is at most one duplication per equivalence class, and $|\kappa| \leq$ the number of distinct $\overline{N_\alpha}$ plus the number of equivalence classes, and both of these cardinalities are $< \aleph$. □

Part 3. *Descending chains.*

In this portion of the proof, the $\{N_\alpha\}$ form a descending chain and $\aleph$ is not regular. We know that the cardinalities of $\mathscr{C}_\sim$ and each $\sim$-class are less than $\aleph$. Thus $|\kappa| \leq \aleph$. So now assume that $|\kappa| = \aleph$. By truncating the chain if neseccary, we may even assume that $\kappa =$ (the first ordinal with cardinality) $\aleph$.

Using transfinite induction we define a family $\{K_{\overline{\alpha}} \mid \overline{\alpha} \in \mathscr{C}_\sim\}$ of submodules $K_{\overline{\alpha}}$ of M by setting $K_{\overline{\alpha}}$ equal to a complement of $\left(\bigoplus_{\overline{\beta}<\overline{\alpha}} K_{\overline{\beta}}\right) \oplus N_{\overline{\alpha}}$ in M. Set

$$K = \bigoplus_{\overline{\alpha}\in\mathscr{C}_\sim} K_{\overline{\alpha}}.$$

Clearly each $N_\alpha + K$ is essential in M. Then $\{N_\alpha + K \mid \alpha < \kappa\}$ has fewer than $\aleph$ distinct elements. Moreover, $(N_\alpha \cap K) = (N_{\alpha+1} \cap K) \Longrightarrow (N_\alpha + K) \neq (N_{\alpha+1} + K)$ by Lemma 3. By omitting the largest element (if it exists) of each subset of κ on which $N_\alpha + K = N_\beta + K$, we get $\{K \cap N_\alpha \mid \alpha \in \kappa\}$ must have $\aleph$ distinct elements and so can be order indexed by κ. By Lemma 1, K has the descending $\aleph$-chain condition. Thus adapting our original notation to K and $\{N_\alpha \cap K\}$, we may assume that

$$M = \bigoplus_{\overline{\alpha}\in\mathscr{C}_\sim} K_{\overline{\alpha}} \quad \text{and} \quad N_{\overline{\alpha}} \cap \left(\bigoplus_{\overline{\beta}\leq\overline{\alpha}} K_{\overline{\beta}}\right) = 0.$$

We will adapt the idea used in proving Theorem 2 to this situation. We find a direct sum of submodules of M each summand of which contains a descending chain of submodules with an essential intersection. When the chains are inserted between intersections in preceeding summands and largest members in following summands, we get a chain of cardinality $\aleph$.

Let us introduce some terminology. A *tail* of κ is a subset $\{\beta\} \subseteq \kappa$ where β runs over the set of upper bounds of some bounded subset of $\mathscr{C}_\sim$. A *finite subsum* of M is a sum $\bigoplus_{\overline{\beta}\in S} K_{\overline{\beta}}$ where S is a finite subset of $\mathscr{C}_\sim$. We observe that any tail has cardinality $\aleph$ and there are precisely $|\mathscr{C}_\sim|$ finite subsums.

Now $\aleph$ is not regular, so we may find a chain $\{\lambda_\sigma \mid \sigma < \text{cofinality}(\aleph)\}$ of cardinals with supremum $\aleph$ where for each σ, $|\mathscr{C}_\sim| < \lambda_\sigma < \aleph$. We will now do a transfinite induction on cofinality($\aleph$) using this notation.

Assume for every $\tau < \sigma$ we have a finite subset S_τ of $\mathscr{C}_\sim$ such that

(i) The finite subsum corresponding to S_τ contains a descending chain of order type $> \lambda_\tau$ consisting of submodules each of which is essential in the largest member of the chain, and

(ii) For all $\upsilon < \tau$, every element of S_τ is greater than every element of S_υ.

We show how to find S_σ preserving these properties. The set $S = \bigcup_{\tau<\sigma} S_\tau$ has cardinality $\leq \aleph_0 \cdot |\sigma| <$ cofinality($\aleph$). Since no equivalence class has $\aleph$

elements, $\mathscr{C}_\sim$ must be cofinal in $\kappa = \aleph$. Thus S is not cofinal in $\mathscr{C}_\sim$. The tail corresponding to S has cardinality $\aleph$, so there is some $\overline{\alpha_\sigma}$ in this tail with $|\{\beta \sim \overline{\alpha_\sigma}\}| > \lambda_\sigma$. For $\beta \sim \overline{\alpha_\sigma}$, let $\overline{N}_\beta$ denote the projection of N_β to $\bigoplus_{\overline{\gamma} > \overline{\alpha_\sigma}} K_{\overline{\gamma}}$. Note that this projection is one-to-one on $N_{\overline{\alpha_\sigma}}$ and so preserves the distinctness of the N_β for $\beta \sim \overline{\alpha_\sigma}$.

For every $\beta \sim \overline{\alpha_\sigma}$, let H_β be a finitely generated submodule of $\overline{N}_\beta$ with H_β not contained in $\overline{N}_{\beta+1}$. This is the categorical equivalent of taking an $x \in \overline{N}_\beta \setminus \overline{N}_{\beta+1}$. By exact direct limits, each H_β is in a finite subsum corresponding to a finite subset of $\{\overline{\gamma} \in \mathscr{C}_\sim \mid \overline{\gamma} > \overline{\alpha_\sigma}\}$. There are at most $|\mathscr{C}_\sim|$ such finite subsums and more than $\lambda_\sigma > |\mathscr{C}_\sim|$ of the H_β, so not every finite subsum can contain $\leq \lambda_\sigma$ of the H_β. We thus have a finite set S_σ of elements of $\mathscr{C}_\sim$ which are all greater than $\overline{\alpha_\sigma}$ and thus upper bounds of $\bigcup_{\tau<\sigma} S_\tau$ with the property that the finite subsum $L_\sigma = \bigoplus_{\overline{\beta} \in S_\sigma} K_{\overline{\beta}}$ contains more than λ_σ of the H_β. Since $\beta \sim \overline{\alpha_\sigma}$, $(\overline{N}_\beta \cap L_\sigma) \subseteq' (\overline{N}_{\overline{\alpha_\sigma}} \cap L_\sigma)$. But $H_\beta \subseteq L_\sigma \Longrightarrow \overline{N}_\beta \cap L_\sigma$ properly contains $\overline{N}_{\beta+1} \cap L_\sigma$, so S_σ satisfies (i) and (ii).

Let H_σ denote a complement of $\overline{N}_{\overline{\alpha_\sigma}} \cap L_\sigma$ in L_σ and set

$$H = \bigoplus_{\sigma < \text{cofinality}(\aleph)} H_\sigma .$$

Let B_σ be the intersection of the first λ_σ distinct $\overline{N}_\beta \cap L_\sigma$ with $\beta \sim \overline{\alpha_\sigma}$. Then $H_\sigma \oplus B_\sigma \subseteq' L_\sigma$.

Then the set

$$\left\{ H \oplus \sum_{\tau<\sigma} B_\tau \oplus (B_\sigma + \overline{N}_\beta \cap L_\sigma) \oplus \sum_{\tau>\sigma} (\overline{N}_{\overline{\alpha_\tau}} \cap L_\tau) \;\middle|\; \sigma < \text{cofinality}(\aleph),\ \beta \sim \overline{\alpha_\sigma} \right\}$$

is a nonascending chain of essential submodules of $\bigoplus_{\sigma<\text{cofinality}(\aleph)} L_\sigma$ which, after duplications are eliminated, can be indexed by an ordinal with cardinality $\geq$ every λ_σ. Such an ordinal has cardinality at least $\aleph$, so this gives us our long awaited contradiction. □

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