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THE NONLOCAL NATURE OF THE SUMMABILITY OF FOURIER SERIES BY CERTAIN ABSOLUTE RIESZ METHODS

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ABSTRACT. It is proved that for a large class of sequences $\{\lambda_n\}$ the summability at a point of a Fourier series $\sum A_n(t)$ by the absolute Riesz method $|R, \lambda_n, 1|$ is not a local property of the generating function. It is also proved, inter alia, that, for every $\varepsilon > 0$, the $|R, \lambda_n, 1|$ summability of the factored series $\sum A_n(t)\lambda_n^{-\varepsilon}$ at any point is always a local property of the generating function.

1. INTRODUCTION

Suppose throughout that, for $n = 1, 2, \ldots$,

$$\mu_n > 0, \qquad \lambda_n := \mu_1 + \mu_2 + \cdots + \mu_n \to \infty,$$

and $s_n := a_1 + a_2 + \cdots + a_n$. The series $\sum a_n$ is said to be summable by the absolute Riesz method $|R, \lambda_n, 1|$ if

$$c(w) := \frac{1}{w} \sum_{\lambda_n < w} (w - \lambda_n) a_n$$

is of bounded variation over (λ_1, ∞), and it is said to be summable by the absolute weighted mean method $|M, \mu_n|$ if the sequence of means $\{t_n\}$ defined by

$$t_n := \frac{1}{\lambda_n} \sum_{\nu=1}^{n} \mu_\nu s_\nu$$

is of bounded variation, that is if

$$\sum_{n=1}^{\infty} |\Delta t_n| < \infty,$$

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where $\Delta t_n := t_n - t_{n+1}$. It is well known, and easily verified, that these two methods are equivalent.

Let

$$\frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} A_n(t) := \frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty}(\alpha_n \cos nt + \beta_n \sin nt)$$

be the Fourier series generated by a periodic function F with period 2π which is Lebesgue integrable over $(-\pi, \pi)$. It is familiar that the convergence of the Fourier series at $t = x$ is a local property of F (i.e. depends only on the behaviour of F in an arbitrarily small neighbourhood of x), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of F. On the other hand, Bosanquet and Kestleman [3] showed that the summability $|C, 1|$ $(= |M, 1|)$ of the Fourier series at any point is not a local property of F, and Mohanty [7] subsequently showed that this is also the case with summability $|R, \lambda_n, 1|$ when $\lambda_n := \ell_k(n)$ for n sufficiently large, where

$$\ell_0(x) := x \quad \text{and} \quad \ell_k(x) := \log(\ell_{k-1}(x))$$

for $k = 1, 2, \ldots$ and x sufficiently large. Mohanty also showed that the $|R, \log n, 1|$ summability of the factored Fourier series

$$\sum_{n=2}^{\infty} A_n(t)/\log n$$

at any point is a local property of F, whereas the $|C, 1|$ summability of this series is not. Matsumoto [5] improved the first of these results by showing that the $|R, \log n, 1|$ summability of the series

$$\sum_{n=3}^{\infty} A_n(t)(\log\log n)^{-p}, \qquad p > 1,$$

at any point is a local property of F, and Bhatt [1] went a step further by showing that the factor $(\log\log n)^{-p}$ in the above series can be replaced by the more general factor $\gamma_n \log n$ where $\{\gamma_n\}$ is a convex sequence such that $\sum \gamma_n/n$ is convergent. Mishra [6] proved that if $\{\gamma_n\}$ is as above, and if

$$\lambda_n = O(n\mu_n) \quad \text{and} \quad \lambda_n \Delta\mu_n = O(\mu_n \mu_{n+1}),$$

then the summability $|M, \mu_n|$ of the series

$$\sum_{n=1}^{\infty} A_n(t)\gamma_n \frac{\lambda_n}{n\mu_n}$$

at any point is a local property of F. This does not directly generalize any of the above-mentioned results involving $|R, \log n, 1|$ summability since the order relations are not satisfied by $\mu_n := 1/n$. Bor [2] recently showed that $|M, \mu_n|$ in Mishra's result can be replaced by a more general summability method $|M, \mu_n|_k$. The object of this paper is to prove the following two theorems which include most of the above-mentioned results as special cases.

Theorem 1. *Suppose that a is a positive integer, and that f is a positive, unbounded function with an absolutely continuous positive derivative on $[e^a, \infty)$ such that, on this interval,*

$$\frac{xf'(x)}{f(x)} \text{ decreases to } 0 \tag{1}$$

and

$$xf''(x) = O(f'(x)). \tag{2}$$

Suppose also that

$$\lambda_n := f(e^n) \quad \text{for } +n \geq a, \tag{3}$$

and that $0 < \alpha < \beta < 2\pi$. Then there is a function F, Lebesgue integrable over (α, β) and zero in the remainder of $(0, 2\pi)$, whose Fourier series is not summable $|R, \lambda_n, 1|$ at $t = 0$.

This shows that, subject to the hypotheses of the theorem, the summability $|R, \lambda_n, 1|$ of a Fourier series at any point is not a local property of its generating function. Since the hypotheses are satisfied by $f(x) := \ell_k(x)$ for $k = 1, 2, \ldots$, Bosanquet and Kestleman's result, and also Mohanty's result, on the nonlocal nature of the summability of a Fourier series by certain absolute methods are special cases of Theorem 1.

Theorem 2. *Suppose that the sequence $\{c_n\}$ is such that*

$$\sum_{n=1}^{\infty} \frac{\mu_n}{\lambda_n} |c_n| < \infty \tag{4}$$

and

$$\sum_{n=1}^{\infty} |\Delta c_n| < \infty. \tag{5}$$

Then the summability $|R, \lambda_n, 1|$ of the factored Fourier series

$$\sum_{n=1}^{\infty} A_n(t) c_n$$

at any point is a local property of the generating function F.

This theorem generalizes Bhatt's above-mentioned result, since it is known (see [1] for references) that if $\{\gamma_n\}$ is a convex sequence such that $\sum \gamma_n / n$ is convergent, then

$$\gamma_n \geq \gamma_{n+1} \geq 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \log n \, \Delta\gamma_n < \infty,$$

and so (4) and (5) are satisfied by $\mu_n := 1/n$, $c_n := \gamma_n \log n$. Since, by Dini's theorem, $\sum \mu_n \lambda_n^{-1-\varepsilon}$ is convergent whenever $\varepsilon > 0$, we have the following corollary of Theorem 2.

Corollary. *For $\varepsilon > 0$, the summability $|R, \lambda_n, 1|$ of the factored Fourier series*

$$\sum_{n=1}^{\infty} A_n(t)\lambda_n^{-\varepsilon}$$

at any point is a local property of the generating function F.

2. Preliminary results

Lemma 1. *Suppose that the function f satisfies the conditions of Theorem 1 and that g is its inverse function. Let $\lambda_n := f(e^n)$, and*

$$h(w) := \frac{g(w)}{wg'(w)} \quad \text{on } +[b, \infty),$$

where $b := f(e^a)$. Then

$$h(w) \text{ decreases to } 0 \tag{6}$$

and

$$wh'(w) = O(1) \tag{7}$$

on $[b, \infty)$. Further,

$$\sum_{n=a}^{\infty} h(\lambda_n) = \infty. \tag{8}$$

Finally, if $\sum a_n$ is summable $|R, \lambda_n, 1|$, then $\sum a_n h(\lambda_n)$ is absolutely convergent.

Proof. Let $w = f(x)$ for $x \geq e^a$. Then $x = g(w)$ and $1 = g'(w)f'(x)$, whence $h(w) = xf'(x)/f(x)$. Thus (6) is a consequence of (1). Next,

$$0 = g''(w)f'(x)^2 + g'(w)f''(x) = \frac{g''(w)}{g'(w)}f'(x) + g'(w)f''(x)$$

so that

$$\begin{aligned} wh'(w) &= wh(w)\left(\frac{g'(w)}{g(w)} - \frac{1}{w} - \frac{g''(w)}{g'(w)}\right) = 1 - h(w) - \frac{wg(w)g''(w)}{g'(w)^2} \\ &= 1 - h(w) + \frac{xf''(x)}{f'(x)}. \end{aligned}$$

Hence (7) is a consequence of (2) and (6).

In order to establish (8), let $\lambda(x) := f(e^x)$, so that $\lambda_n = \lambda(n)$. Then, for $x \geq a$, we have $g(\lambda(x)) = e^x$ so that $g'(\lambda(x))\lambda'(x) = e^x$, and hence

$$h(\lambda(x)) = \frac{g(\lambda(x))}{\lambda(x)g'(\lambda(x))} = \frac{\lambda'(x)}{\lambda(x)}.$$

Therefore

$$\int_a^y h(\lambda(x))\,dx = \log(\lambda(y)) - \log(\lambda(a)) \to \infty \quad \text{as } y \to \infty.$$

Conclusion (8) follows, by the integral test.

Suppose now that $\sum a_n$ is $|R, \lambda_n, 1|$ summable. Since $g(\lambda_n) = e^n$, it follows from (6) and (7), by a result due to Dikshit [4], that $\sum a_n h(\lambda_n)$ is

$|R, e^n, 1|$ summable, and Mohanty [8, Lemma 4] has shown this to be equivalent to $\sum a_n h(\lambda_n)$ being absolutely convergent. □

Lemma 2. *Suppose that the sequence* $\{c_n\}$ *satisfies conditions* (4) *and* (5) *of Theorem* 2, *and that* $\{s_n\}$ *is bounded. Then*

$$\sum_{n=1}^{\infty} a_n c_n \tag{9}$$

is summable $|R, \lambda_n, 1|$.

Proof. Let $\{T_n\}$ be the sequence of (M, μ_n) means of series (9), that is

$$T_n := \frac{1}{\lambda_n}\sum_{\nu=1}^{n}\mu_\nu \sum_{r=1}^{\nu} a_r c_r = \frac{1}{\lambda_n}\sum_{r=1}^{n}(\lambda_n - \lambda_{r-1}) a_r c_r$$

where $\lambda_0 := 0$. We wish to show that

$$\sum_{n=1}^{\infty} |\Delta t_n| < \infty.$$

We have that

$$\begin{aligned} T_{n+1} - T_n &= \frac{\mu_{n+1}}{\lambda_n \lambda_{n+1}} \sum_{r=1}^{n+1} \lambda_{r-1} a_r c_r \\ &= \frac{\mu_{n+1}}{\lambda_n \lambda_{n+1}} \sum_{r=1}^{n} s_r(\lambda_r \Delta c_r - \mu_r c_r) + \frac{\mu_{n+1} c_{n+1} s_{n+1}}{\lambda_{n+1}}. \end{aligned}$$

Hence, if we suppose that $|s_n| \leq 1$, as we may without loss of generality, we see that

$$|\Delta t_n| \leq \frac{\mu_{n+1}}{\lambda_n \lambda_{n+1}} \sum_{r=1}^{n} (\lambda_r |\Delta c_r| + \mu_r |c_r|) + \frac{\mu_{n+1}}{\lambda_{n+1}} |c_{n+1}|,$$

and so

$$\begin{aligned} \sum_{n=1}^{\infty} |\Delta t_n| &\leq \sum_{r=1}^{\infty} (\lambda_r |\Delta c_r| + \mu_r |c_r|) \sum_{n=r}^{\infty} \frac{\mu_{n+1}}{\lambda_n \lambda_{n+1}} + \sum_{n=2}^{\infty} \frac{\mu_n}{\lambda_n} |c_n| \\ &= \sum_{r=1}^{\infty} \left(|\Delta c_r| + \frac{\mu_r}{\lambda_r} |c_r| \right) + \sum_{n=2}^{\infty} \frac{\mu_n}{\lambda_n} |c_n| < \infty, \end{aligned}$$

by (4) and (5). □

3. Proofs of the theorems

Proof of Theorem 1. By Lemma 1, it suffices to show that there is a function F, Lebesgue integrable over (α, β), such that

$$\sum_{n=a}^{\infty} \left| \int_\alpha^\beta h(\lambda_n) F(t) \cos nt \, dt \right| = \infty,$$

where the function h is as in the lemma. For $0 < t < 2\pi$, $t \neq \pi$, we have that

$$\sum_{n=a}^{\infty} h(\lambda_n)|\cos nt| \geq \sum_{n=a}^{\infty} h(\lambda_n)\cos^2 nt$$
$$\geq \frac{1}{2}\sum_{n=a}^{\infty} h(\lambda_n) - \frac{1}{2}\left|\sum_{n=a}^{\infty} h(\lambda_n)\cos 2nt\right| = \infty,$$

by Lemma 1, the final sum being convergent because the sequence $\{h(\lambda_n)\}$ decreases to 0. The required result now follows from a theorem due to Bosanquet and Kestleman [3, Theorem 1]. □

Proof of Theorem 2. Since the convergence of the Fourier series at a point is a local property of its generating function F, Theorem 2 follows immediately from Lemma 2. □

Remark (added November 9, 1990). After this paper was accepted for publication I found out that Theorem 2 is in fact a special case of Theorem 3 in S. Baron's paper, *Local property of absolute summability of a Fourier series and the conjugate series*, Tartu Riikl. Ül. Toimetised Vih. **253** (1970), 212–228. My proof, however, is somewhat simpler and more direct than Baron's, partly because he deals with more general summability methods.

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