THE NONLOCAL NATURE OF THE SUMMABILITY OF FOURIER SERIES BY CERTAIN ABSOLUTE RIESZ METHODS

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Abstract. It is proved that for a large class of sequences \( \{\lambda_n\} \) the summability at a point of a Fourier series \( \sum A_n(t) \) by the absolute Riesz method \( |R, \lambda_n, 1| \) is not a local property of the generating function. It is also proved, inter alia, that, for every \( \epsilon > 0 \), the \( |R, \lambda_n, 1| \) summability of the factored series \( \sum A_n(t)\lambda_n^{-\epsilon} \) at any point is always a local property of the generating function.

1. Introduction

Suppose throughout that, for \( n = 1, 2, \ldots \),
\[
\mu_n > 0, \quad \lambda_n := \mu_1 + \mu_2 + \cdots + \mu_n \to \infty,
\]
and \( s_n := a_1 + a_2 + \cdots + a_n \). The series \( \sum a_n \) is said to be summable by the absolute Riesz method \( |R, \lambda_n, 1| \) if
\[
c(w) := \frac{1}{w} \sum_{\lambda_n < w} (w - \lambda_n)a_n
\]
is of bounded variation over \( (\lambda_1, \infty) \), and it is said to be summable by the absolute weighted mean method \( |M, \mu_n| \) if the sequence of means \( \{t_n\} \) defined by
\[
t_n := \frac{1}{\lambda_n} \sum_{\nu=1}^{n} \mu_\nu s_\nu
\]
is of bounded variation, that is if
\[
\sum_{n=1}^{\infty} |\Delta t_n| < \infty,
\]

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where $\Delta t_n := t_n - t_{n+1}$. It is well known, and easily verified, that these two methods are equivalent.

Let

$$\frac{1}{2} \alpha_0 + \sum_{n=1}^{\infty} A_n(t) := \frac{1}{2} \alpha_0 + \sum_{n=1}^{\infty} (\alpha_n \cos nt + \beta_n \sin nt)$$

be the Fourier series generated by a periodic function $F$ with period $2\pi$ which is Lebesgue integrable over $(-\pi, \pi)$. It is familiar that the convergence of the Fourier series at $t = x$ is a local property of $F$ (i.e. depends only on the behaviour of $F$ in an arbitrarily small neighbourhood of $x$), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of $F$. On the other hand, Bosanquet and Kestleman [3] showed that the summability $|C, 1| (= |M, 1|)$ of the Fourier series at any point is not a local property of $F$, and Mohanty [7] subsequently showed that this is also the case with summability $|R, \lambda_n, 1|$ when $\lambda_n := \ell_k(n)$ for $n$ sufficiently large, where

$$\ell_0(x) := x \quad \text{and} \quad \ell_k(x) := \log(\ell_{k-1}(x))$$

for $k = 1, 2, \ldots$ and $x$ sufficiently large. Mohanty also showed that the $|R, \log n, 1|$ summability of the factored Fourier series

$$\sum_{n=2}^{\infty} A_n(t)/ \log n$$

at any point is a local property of $F$, whereas the $|C, 1|$ summability of this series is not. Matsumoto [5] improved the first of these results by showing that the $|R, \log n, 1|$ summability of the series

$$\sum_{n=3}^{\infty} A_n(t)(\log \log n)^{-p}, \quad p > 1,$$

at any point is a local property of $F$, and Bhatt [1] went a step further by showing that the factor $(\log \log n)^{-p}$ in the above series can be replaced by the more general factor $\gamma_n \log n$ where $\{\gamma_n\}$ is a convex sequence such that $\sum \gamma_n/n$ is convergent. Mishra [6] proved that if $\{\gamma_n\}$ is as above, and if

$$\lambda_n = O(n\mu_n) \quad \text{and} \quad \lambda_n \Delta \mu_n = O(\mu_n\mu_{n+1}),$$

then the summability $|M, \mu_n|$ of the series

$$\sum_{n=1}^{\infty} A_n(t)\gamma_n \frac{\lambda_n}{n\mu_n}$$

at any point is a local property of $F$. This does not directly generalize any of the above-mentioned results involving $|R, \log n, 1|$ summability since the order relations are not satisfied by $\mu_n := 1/n$. Bor [2] recently showed that $|M, \mu_n|$ in Mishra’s result can be replaced by a more general summability method $|M, \mu_n|_k$. The object of this paper is to prove the following two theorems which include most of the above-mentioned results as special cases.
Theorem 1. Suppose that $a$ is a positive integer, and that $f$ is a positive, unbounded function with an absolutely continuous positive derivative on $[e^a, \infty)$ such that, on this interval,

\[ \frac{xf'(x)}{f(x)} \text{ decreases to } 0 \]

and

\[ xf''(x) = O(f'(x)). \]

Suppose also that

\[ \sum_{n=a}^{\infty} x^n = f(x^n) \text{ for } x > e^a, \]

and that $0 < \alpha < \beta < 2\pi$. Then there is a function $F$, Lebesgue integrable over $(\alpha, \beta)$ and zero in the remainder of $(0, 2\pi)$, whose Fourier series is not summable $|R, \lambda_n, 1|$ at $t = 0$.

This shows that, subject to the hypotheses of the theorem, the summability $|R, \lambda_n, 1|$ of a Fourier series at any point is not a local property of its generating function. Since the hypotheses are satisfied by $f(x) := \delta_k(x)$ for $k = 1, 2, \ldots$, Bosanquet and Kestleman's result, and also Mohanty's result, on the nonlocal nature of the summability of a Fourier series by certain absolute methods are special cases of Theorem 1.

Theorem 2. Suppose that the sequence $\{c_n\}$ is such that

\[ \sum_{n=1}^{\infty} \frac{\mu_n}{\lambda_n} |c_n| < \infty \]

and

\[ \sum_{n=1}^{\infty} |\Delta c_n| < \infty. \]

Then the summability $|R, \lambda_n, 1|$ of the factored Fourier series

\[ \sum_{n=1}^{\infty} A_n(t)c_n \]

at any point is a local property of the generating function $F$.

This theorem generalizes Bhatt's above-mentioned result, since it is known (see [1] for references) that if $\{\gamma_n\}$ is a convex sequence such that $\sum \gamma_n/n$ is convergent, then

\[ \gamma_n \geq \gamma_{n+1} \geq 0 \text{ and } \sum_{n=1}^{\infty} \log n \Delta \gamma_n < \infty, \]

and so (4) and (5) are satisfied by $\mu_n := 1/n$, $\lambda_n := \gamma_n \log(n)$. Since, by Dini's theorem, $\sum \mu_n \lambda_n^{-1-\varepsilon}$ is convergent whenever $\varepsilon > 0$, we have the following corollary of Theorem 2.
Corollary. For \( \varepsilon > 0 \), the summability \(|R, \lambda_n, 1|\) of the factored Fourier series
\[
\sum_{n=1}^{\infty} A_n(t) \lambda_n^{-\varepsilon}
\]
at any point is a local property of the generating function \( F \).

2. Preliminary results

Lemma 1. Suppose that the function \( f \) satisfies the conditions of Theorem 1 and that \( g \) is its inverse function. Let \( \lambda_n := f(e^n) \), and
\[
h(w) := \frac{g(w)}{w g'(w)} \quad \text{on} \quad [b, \infty),
\]
where \( b := f(h_a) \). Then
\( h(w) \) decreases to 0
and
\( wh'(w) = O(1) \)
on \([b, \infty)\). Further,
\( \sum_{n=a}^{\infty} h(\lambda_n) = \infty. \)

Finally, if \( \sum a_n \) is summable \(|R, \lambda_n, 1|\), then \( \sum a_n h(\lambda_n) \) is absolutely convergent.

Proof. Let \( w = f(x) \) for \( x \geq e^a \). Then \( x = g(w) \) and \( 1 = g'(w)f'(x) \), whence \( h(w) = xf'(x)/f(x) \). Thus (6) is a consequence of (1). Next,
\[
0 = g''(w)f'(x)^2 + g'(w)f''(x) = \frac{g''(w)}{g'(w)} f'(x) + g'(w)f''(x)
\]
so that
\[
wh'(w) = wh(w) \left( \frac{g'(w)}{g(w)} - \frac{1}{w} - \frac{g''(w)}{g'(w)} \right) = 1 - h(w) - \frac{wg(w)g''(w)}{g'(w)^2}
\]
\[
= 1 - h(w) + \frac{xf''(x)}{f'(x)}.
\]
Hence (7) is a consequence of (2) and (6).

In order to establish (8), let \( \lambda(x) := f(e^x) \), so that \( \lambda_n = \lambda(n) \). Then, for \( x \geq a \), we have \( g(\lambda(x)) = e^x \) so that \( g'(\lambda(x))\lambda'(x) = e^x \), and hence
\[
h(\lambda(x)) = \frac{g(\lambda(x))}{\lambda(x) g'(\lambda(x))} = \frac{\lambda'(x)}{\lambda(x)}.
\]
Therefore
\[
\int_a^y h(\lambda(x)) \, dx = \log(\lambda(y)) - \log(\lambda(a)) \to \infty \quad \text{as} \quad y \to \infty.
\]

Conclusion (8) follows, by the integral test.

Suppose now that \( \sum a_n \) is \(|R, \lambda_n, 1|\) summable. Since \( g(\lambda_n) = e^n \), it follows from (6) and (7), by a result due to Dikshit [4], that \( \sum a_n h(\lambda_n) \) is
\[ |R, e^n, 1| \text{ summable, and Mohanty} \ [8, \text{Lemma 4}] \text{ has shown this to be equivalent to } \sum a_n h(\lambda_n) \text{ being absolutely convergent.} \]

**Lemma 2.** Suppose that the sequence \( \{c_n\} \) satisfies conditions (4) and (5) of Theorem 2, and that \( \{s_n\} \) is bounded. Then

\[
\sum_{n=1}^{\infty} a_n c_n
\]

is summable \( |R, \lambda_n, 1| \).

**Proof.** Let \( \{T_n\} \) be the sequence of \( (M, \mu_n) \) means of series (9), that is

\[
T_n := \frac{1}{\lambda_n} \sum_{\nu=1}^{n} \mu_\nu \sum_{r=1}^{\nu} a_r c_r = \frac{1}{\lambda_n} \sum_{r=1}^{n} (\lambda_n - \lambda_{r-1}) a_r c_r
\]

where \( \lambda_0 := 0 \). We wish to show that

\[
\sum_{n=1}^{\infty} |\Delta T_n| < \infty.
\]

We have that

\[
T_{n+1} - T_n = \frac{\mu_{n+1}}{\lambda_n \lambda_{n+1}} \sum_{r=1}^{n} \lambda_{r-1} a_r c_r
\]

Hence, if we suppose that \( |s_n| \leq 1 \), as we may without loss of generality, we see that

\[
|\Delta T_n| \leq \frac{\mu_{n+1}}{\lambda_n \lambda_{n+1}} \sum_{r=1}^{n} (\lambda_r |\Delta c_r| + \mu_r |c_r|) + \frac{\mu_{n+1} c_{n+1}}{\lambda_{n+1}} |c_{n+1}|,
\]

and so

\[
\sum_{n=1}^{\infty} |\Delta T_n| \leq \sum_{r=1}^{\infty} (\lambda_r |\Delta c_r| + \mu_r |c_r|) \sum_{n=r}^{\infty} \frac{\mu_{n+1}}{\lambda_n \lambda_{n+1}} + \sum_{n=2}^{\infty} \frac{\mu_n}{\lambda_n} |c_n|
\]

\[
= \sum_{r=1}^{\infty} \left( |\Delta c_r| + \frac{\mu_r}{\lambda_r} |c_r| \right) + \sum_{n=2}^{\infty} \frac{\mu_n}{\lambda_n} |c_n| < \infty,
\]

by (4) and (5). \( \square \)

### 3. Proofs of the theorems

**Proof of Theorem 1.** By Lemma 1, it suffices to show that there is a function \( F \), Lebesgue integrable over \( (\alpha, \beta) \), such that

\[
\sum_{n=a}^{\infty} \left| \int_{\alpha}^{\beta} h(\lambda_n) F(t) \cos nt \, dt \right| = \infty,
\]
where the function $h$ is as in the lemma. For $0 < t < 2\pi$, $t \neq \pi$, we have that
\[
\sum_{n=a}^{\infty} h(\lambda_n) \cos nt \geq \sum_{n=a}^{\infty} h(\lambda_n) \cos^2 nt
\]
\[
\geq \frac{1}{2} \sum_{n=a}^{\infty} h(\lambda_n) - \frac{1}{2} \left| \sum_{n=a}^{\infty} h(\lambda_n) \cos 2nt \right| = \infty,
\]
by Lemma 1, the final sum being convergent because the sequence \{h(\lambda_n)\} decreases to 0. The required result now follows from a theorem due to Bosanquet and Kestleman [3, Theorem 1].

**Proof of Theorem 2.** Since the convergence of the Fourier series at a point is a local property of its generating function $F$, Theorem 2 follows immediately from Lemma 2. □

**Remark** (added November 9, 1990). After this paper was accepted for publication I found out that Theorem 2 is in fact a special case of Theorem 3 in S. Baron's paper, *Local property of absolute summability of a Fourier series and the conjugate series*, Tartu Riikl. Ü1. Toimetised Vih. 253 (1970), 212–228. My proof, however, is somewhat simpler and more direct than Baron’s, partly because he deals with more general summability methods.

**References**


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