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LINDELÖF SPACES CONCENTRATED ON BERNSTEIN SUBSETS OF THE REAL LINE

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ABSTRACT. We show in ZFC that for each n with $n \in \omega$ or $n = \omega$, there is a Lindelöf space X and a separable metric space M such that for every $m < n$, $X \times {}^m M$ is Lindelöf, whereas $X \times {}^n M$ is nonnormal.

0. INTRODUCTION

History. In 1975 [9] M. E. Rudin and M. Starbird proved the following theorem: Suppose X, M, and C are Hausdorff completely regular spaces with M metrizable and C compact. Then the product space $X \times M \times C$ is normal if each of the subproducts $X \times M$ and $X \times C$ is normal. In his Ph.D. thesis, Starbird asked if this implication still holds when we replace C by M (alternatively, replace M by C). The question was raised again by T. C. Przymusiński in his Handbook article [8]. As we will show, the answer is yes if M is separable and complete; but the implication fails when completeness is dropped from the hypothesis. All results are in ZFC.

First, suppose M is separable completely metrizable and $X \times M$ is normal. We will use two cases to show that $X \times {}^2M$ is also normal. Suppose M is σ-compact. We can assume that M is nondiscrete and therefore contains a convergent sequence, in which case normality in $X \times M$ implies not only that X is normal but also countably paracompact (C. H. Dowker, 1951; see [8, p. 794]). The product of a countably paracompact normal space and a σ-compact metric space is normal (K. Morita, 1963; see [8, p. 807]). The second case follows immediately from a standard theorem: If Y is separable completely metrizable and not σ-compact, then $X \times Y$ is normal iff $X \times \mathbb{P}$ is normal, where $\mathbb{P}$ denotes the irrationals with the usual topology (see [3, Proposition 1]). Apply this theorem twice: with $Y = M$ and $Y = {}^2M$.

Our answer to Starbird's question for noncomplete spaces begins with a classical example, due to E. A. Michael, of a Lindelöf space and a separable metric

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space with a nonnormal product (1963, [4]). Let $\mathbb{R}$ denote the real line. In Michael's example, the Lindelöf space is the refinement of $\mathbb{R}$ obtained by isolating each point in the complement of an arbitrarily chosen Bernstein subset A; and the metric space is $\mathbb{R}\backslash A$ with the subspace topology. A subset of an uncountable separable completely metrizable space is Bernstein if both it and its complement intersect every uncountable closed set (a set of this type in the real line was first constructed by F. Bernstein in 1908—see [6, pp. 23–24]). The Lindelöf property in the nonmetrizable factor follows from the concentration of $\mathbb{R}$ on A. A topological space X is concentrated on a subset A if for each open set $U \supseteq A$, $X\backslash U$ is countable. Note that $\mathbb{R}$ is concentrated on $A \subseteq \mathbb{R}$ iff A contains a Bernstein set.

In 1980 Przymusiński gave an example of a Lindelöf space X such that ${}^nX$ is Lindelöf for every $n \in \omega$, whereas ${}^\omega X$ is nonnormal [7]. (An example of this type was given under the Continuum Hypothesis by Michael in 1971 [5].) In his Handbook article [1, pp. 395–397], D. K. Burke describes a variation of this example using Michael's idea of isolating points off a Bernstein set. In the current paper we show that Bernstein sets chosen by the Przymusiński–Burke method (which is essentially the same as the construction used in the proof of Lemma 5 below) also yields examples as described in the abstract. So the answer is no to Starbird's question: Is $X \times {}^2M$ normal if M is metric and $X \times M$ is normal? There is a Bernstein set $A \subseteq \mathbb{R}$ such that for X defined by isolating off A and M taken to be A with the subspace topology from $\mathbb{R}$, $X \times M$ is Lindelöf, whereas $X \times {}^2M$ is nonnormal. (Note that in contrast to Michael's example, the metric space is A with the subspace topology rather than $\mathbb{R}\backslash A$. See [3, Remark on p. 540].) A necessary and sufficient condition on the choice of A is that the Lindelöf property hold in ${}^2X$ and fail in ${}^3X$.

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Notation. Let $\mathbb{R}$ and $\mathbb{C}$ denote the real line and the Cantor ternary set respectively with their usual topologies. For a set Y, $|Y|$ denotes the cardinality of Y. Let $\underline{c} = |\mathbb{R}|$. For each Bernstein set $A \subseteq \mathbb{R}$, let $\mathbb{L}(\mathbb{R}, A)$ denote the Lindelöf refinement of $\mathbb{R}$ obtained by isolating each point of $\mathbb{R}\backslash A$. For a function f, $\operatorname{ran} f$ denotes the range of f.

Recall that $\omega = \{0, 1, 2, \ldots\}$, and for $n \in \omega$, $n = \{0, 1, 2, \ldots, n-1\}$. Suppose $n \in \omega$ or $n = \omega$ and $\alpha, \beta \subseteq n$ with $\beta = n\backslash\alpha$. In reference to a product of real lines where n is the index set, define π_β to be the projection function to β, and define a section to be a point-inverse set under a projection (for $\beta = \{i\}$, we will abbreviate π_β by π_i and identify ${}^\beta\mathbb{R}$ with $\mathbb{R}$). A Cantor set $C \subseteq {}^n\mathbb{R}$ is defined to be diagonal over α if it belongs to a section of π_β, and for each $i \in \alpha$, $\pi_i|C$ is 1-1. A Cantor set is diagonal if it is diagonal over some α.

The reader familiar with Przymusiński's work will note that a diagonal set is a variation on his concept of n-cardinality. We found this variation to be somewhat more convenient for presenting our point of view on these examples.

1. Lemmas

Lemma 1 (M. E. Rudin and M. Starbird, [9]). *The product of a Lindelöf space and a separable metric space is normal iff it is Lindelöf.*

Lemma 2. *Suppose $n \in \omega$ and $Y \subseteq {}^n\mathbb{R}$ with $|Y| = \omega_1$. Then there exists $Y' \subseteq Y$, $\alpha \subseteq n$, and a Cantor set C diagonal over α such that if $\beta = n \backslash \alpha$ and T is the section of π_β containing C, then $Y' \subseteq Y \cap T$, $|Y'| = \omega_1$, and every point of C is a condensation point in ${}^n\mathbb{R}$ of Y'. (Recall that in a topological space X, a point $x \in X$ is a condensation point of an uncountable subset $Y \subseteq X$ if for every open set U containing x, $|U \cap Y| \geq \omega_1$.)*

Proof. First, choose $\langle Z_0, \ldots, Z_{n-1} \rangle$ such that $Z_0 \subseteq Y$, and for each $i > 0$, Z_i is an uncountable subset of Z_{i-1} with $\pi_i | Z_i$ either 1-1 or constant. Let $Y' = Z_{n-1}$, and let α be the set of all indices i for which $\pi_i | Y'$ is 1-1. Let $\beta = n \backslash \alpha$ and let T be the section of π_β containing Y'.

Then in the subspace topology of T, let $\langle U_m : m \in \omega \rangle$ be a sequence of basic open sets such that $U_0 = T$ and for each $m > 0$: (1) $\mathrm{cl}(U_m) \subseteq U_{m-1}$; (2) each component of U_{m-1} contains at least two but no more than a finite number of components of U_m; (3) for each $i \in \alpha$, the projections under π_i of the components of U_m are pairwise disjoint open intervals with diameter $< 1/m$; and, (4) each component of U_m has uncountable intersection with Y'. Let $C = \bigcap_{m \in \omega} U_m$. By completeness in T, C is a Cantor set.

Lemma 3. *Suppose $A \subseteq \mathbb{R}$ is a Bernstein set and $X = \mathbb{L}(\mathbb{R}, A)$. Then for each $n \in \omega$, the following are equivalent ((2) through (4) are equivalent for $n = \omega$ also):*

(1) *every closed discrete subset of ${}^nX$ is countable;*
(2) *every diagonal Cantor set in ${}^n\mathbb{R}$ has a limit point in the topology of ${}^nX$;*
(3) *every Cantor set in ${}^n\mathbb{R}$ that is diagonal over $\alpha \subseteq n$ contains a point where every coordinate of its projection to α belongs to A;*
(4) *for each $m \leq n$, every Cantor set in ${}^m\mathbb{R}$ that is diagonal over m intersects ${}^mA$.*

Proof. For $n \in \omega$ or $n = \omega$ the implications $(1) \Rightarrow (2)$ and $(2) \Leftrightarrow (3) \Leftrightarrow (4)$ are immediate from the definitions. For $n \in \omega$, $\neg(1) \Rightarrow \neg(3)$ follows from Lemma 2.

Lemma 4. *Suppose $n \in \omega$ or $n = \omega$, C is a Cantor set in ${}^n\mathbb{R}$ diagonal over n, and $x \in \mathbb{R}$. Then $|\{f \in C : x \in \operatorname{ran} f\}| \leq n$.*

Proof. For each $f \in C$ with $x \in \operatorname{ran} f$, let $\tau(f)$ be an integer $i < n$ with $f(i) = x$. Since $\pi_i | C$ is 1-1 for each $i < n$, τ is also 1-1.

Lemma 5. *For each $n > 1$ with $n \in \omega$ or $n = \omega$, there is a Bernstein set $A \subseteq \mathbb{R}$ and a Cantor set $D \subseteq {}^n\mathbb{R}$ diagonal over n such that for $X = \mathbb{L}(\mathbb{R}, A)$, D is discrete in ${}^nX$, whereas for each $m < n$ and each diagonal Cantor set $C \subseteq {}^m\mathbb{R}$, C has a limit point in ${}^mX$.*

Proof. Let $\mathscr{D} = \{C :$ there exists $m < n$ such that C is a Cantor set in ${}^m\mathbb{R}$ diagonal over $m\}$. First note that by Lemma 3 every diagonal C has a limit point if every set in $\mathscr{D}$ intersects ${}^mA$ for some m.

Let $D = \operatorname{ran} \sigma$ where $\sigma : \mathbb{C} \to {}^n\mathbb{R}$ is defined as follows. For each $m < n$, let h_m be a homeomorphism of $\mathbb{C}$ onto a Cantor set in $\mathbb{R}$ such that $h_i[\mathbb{C}] \cap h_j[\mathbb{C}] = \varnothing$ for $i \neq j$. For each $x \in \mathbb{C}$ and $i < n$, let $\sigma(x)_i = h_i(x)$.

We construct A as follows. Well order each of $\mathbb{R}$, $\mathbb{C}$, and $\mathscr{D}$ in type $\underline{c}$. Let $\prec$ denote the well ordering of $\mathscr{D}$. By Lemma 4, we can define $F : \mathscr{D} \to \mathbb{R}$

and $G: \mathscr{D} \to \bigcup_{m<n} {}^m\mathbb{R}$ by transfinite recursion so that: for each $C \in \mathscr{D}$,

(1) $F(C) \in \operatorname{ran} f$ for some $f \in C$,
(2) $G(C) \in C$, and
(3) $\{F(C') : C' \prec C\} \cap \bigcup\{\operatorname{ran} G(C') : C' \prec C\} = \varnothing$; and for each $x \in \mathbb{C}$,
(4) there is a unique $C \in \mathscr{D}$ such that $F(C) \in \operatorname{ran} \sigma(x)$.

Let $A = \mathbb{R} \setminus \operatorname{ran} F$. Note that $\operatorname{ran} G(C) \subseteq A$ for each $C \in \mathscr{D}$.

By (1) through (3), A is a Bernstein set. By (2) and Lemma 3, for each $m < n$ and diagonal $C \subseteq {}^m\mathbb{R}$, C has a limit point in ${}^mX$. By (4), $\operatorname{ran} \sigma(x) \setminus A \neq \varnothing$ for each $x \in \mathbb{C}$, so again by Lemma 3, D is discrete in ${}^nX$. (The uniqueness in the choice of C in (4) will be used in the proof of Theorem 2.)

2. Theorems

Theorem 1. *Suppose $A \subseteq \mathbb{R}$ is a Bernstein set, $X = \mathbb{L}(\mathbb{R}, A)$, M is A with the subspace topology from $\mathbb{R}$, and $n \in \omega$ with $n > 1$. Then the following are equivalent:*

(1) *${}^nX$ is Lindelöf;*
(2) *${}^nX$ is normal;*
(3) *$X \times {}^{n-1}M$ is normal;*
(4) *every closed discrete subset of ${}^nX$ is countable.*

Proof. The implication $(1) \Rightarrow (2)$ is standard (see [2], p. 247). For $(2) \Rightarrow (3)$, note that $X \times {}^{n-1}M$ is homeomorphic to a closed subspace of ${}^nX$.

We will prove $\neg(4) \Rightarrow \neg(3)$. Let n be the least integer for which ${}^nX$ fails to satisfy (4). Since A is Bernstein, $n > 1$. By Lemma 3, we can choose a diagonal Cantor set $C \subseteq {}^n\mathbb{R}$ discrete in ${}^nX$. Let $\gamma = n \setminus \{0\}$. By the minimality in the choice of n, C is diagonal over n, and every Cantor subset of $\pi_\gamma[C]$ has a limit point in ${}^\gamma X$. Let $C' = \{f \in C : f|\gamma \in {}^\gamma A\}$. Then in ${}^\gamma\mathbb{R}$, $\pi_\gamma[C']$ is a Bernstein subset of $\pi_\gamma[C]$. Thus C' is an uncountable closed discrete set homeomorphic to a closed subspace of $X \times {}^{n-1}M$. The latter space, therefore, cannot be Lindelöf, and in turn by Lemma 1, cannot be normal. (For a direct proof, note that since C' is discrete, $f(0) \in \mathbb{R} \setminus A$ for each $f \in C'$, so C' and ${}^nA$ are disjoint closed sets; by [3, Lemma 2], these two sets cannot be separated.)

For $(4) \Rightarrow (1)$, we will use induction on euclidean dimension to prove that each section is a Lindelöf subspace of ${}^nX$. The one-dimensional case follows from choosing A to be Bernstein. Suppose that each $(n-1)$-dimensional section is Lindelöf. For a given open cover of ${}^nX$, choose a countable subcollection that covers ${}^nA$. We claim that the complement of the union of this subcollection is contained in the union of countably many $(n-1)$-dimensional sections. Otherwise, we can choose a subset Y of the complement such that $|Y| = \omega_1$, and $\pi_i|Y$ is 1-1 for each $i < n$. Let $C \subseteq {}^n\mathbb{R}$ be the Cantor set diagonal over n that we obtain by applying Lemma 2 to Y. Then C is disjoint from ${}^nA$ contradicting Lemma 3.

Theorem 2. *For each n with $n \in \omega$ or $n = \omega$, there is a Lindelöf space X and a separable metric space M such that for each $m < n$, $X \times {}^mM$ is Lindelöf, whereas $X \times {}^nM$ is nonnormal; moreover, we can take X to be $\mathbb{L}(\mathbb{R}, A)$ and M to be A with the subspace topology from $\mathbb{R}$ for a suitable choice of a Bernstein set $A \subseteq \mathbb{R}$.*

Proof. Choose A according to the construction given in the proof of Lemma 5. For $n \in \omega$, the result follows immediately from Lemma 3 and Theorem 1. For $n = \omega$, again use Lemma 3 and Theorem 1 to show that for each $m \in \omega$, $X \times {}^m M$ is Lindelöf. To show that $X \times {}^\omega M$ is nonnormal, note that by condition (1) and the uniqueness requirement in condition (4) in the construction of A, $\{f \in D : f(0) \in \operatorname{ran} F\}$ is homeomorphic to an uncountable closed discrete subspace of $X \times {}^\omega M$. Use Lemma 1 to complete the proof.

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