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A NOTE ON THE CONNECTEDNESS PROBLEM FOR NEST ALGEBRAS

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ABSTRACT. It has been conjectured that a certain operator T belonging to the group $\mathcal{G}$ of invertible elements of the algebra $\operatorname{Alg}\mathbf{Z}$ of doubly infinite upper-triangular bounded matrices lies outside the connected component of the identity in $\mathcal{G}$. In this note we show that T actually lies inside the connected component of the identity of $\mathcal{G}$.

Let $\mathbf{T}$ be the unit circle in the complex plane with normalized Lebesgue measure. For $1 \leq p \leq \infty$, let H^p be the usual Hardy space of all functions in $L^p(\mathbf{T})$ that have analytic extensions to the open unit disk $\mathbf{D}$. Let $\mathscr{H} = L^2(\mathbf{T})$ and let $\mathscr{B}(\mathscr{H})$ be set of all bounded linear operators on $\mathscr{H}$. Let $W \in \mathscr{B}(\mathscr{H})$ be the shift operator: $(Wf)(e^{i\theta}) = e^{i\theta}f(e^{i\theta})$. In this paper, we consider the nest $\{W^nH^2 : n \in \mathbf{Z}\}$ of subspaces of $L^2(\mathbf{T})$, and its associated nest algebra,

$$\operatorname{Alg}\mathbf{Z} = \{T \in \mathscr{B}(\mathscr{H}) : TW^nH^2 \subseteq W^nH^2 \text{ for all } n \in \mathbf{Z}\}.$$

A question which has been unanswered for several years is the following:

Question. Is the group of invertible elements of the Banach algebra $\operatorname{Alg}\mathbf{Z}$ connected in the norm topology?

It is frequently conjectured that the answer to this question is no. The reason for conjecturing a negative answer is because of a strong analogy between nest algebras and analytic function theory. We refer the reader to the book by Davidson [1] for details and more background on this question.

For each $f \in L^\infty(\mathbf{T})$, let $M_f \in \mathscr{B}(\mathscr{H})$ be the multiplication operator,

$$M_f\phi = f\phi, \qquad \phi \in L^2(\mathbf{T}).$$

Note that for $f \in H^\infty$, we have $M_f \in \operatorname{Alg}\mathbf{Z}$. Let a be a positive real number and set

$$h(z) = \frac{ai}{\pi}\log\left(\frac{1+z}{1-z}\right).$$

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Then h is a conformal map of the open unit disk onto the unbounded vertical strip $\{z \in \mathbf{C} : -a < \mathrm{Re}(z) < a\}$.

If $f = \exp(h)$ then it is easy to see that both f and $1/f$ are H^∞ functions and moreover, that f is not the exponential of any H^∞ function. Therefore f cannot be connected to the constant function 1 via a norm continuous path within the group of invertible elements of the Banach algebra H^∞. For this reason, the operator M_f has been suggested as a possible example of an operator which cannot be connected to the identity via a norm continuous path inside the group of invertibles in $\mathrm{Alg}\,\mathbf{Z}$.

The purpose of this note is to show that in fact, M_f *may be connected to the identity via a norm continuous path of invertible elements in* $\mathrm{Alg}\,\mathbf{Z}$.

Before giving the proof we pause for some terminology and to make a few simple remarks.

Let $\mathscr{A}$ be a unital Banach algebra with unit I. Say that an invertible element a of $\mathscr{A}$ may be *connected to the identity* if thee exists a norm continuous function $f: [0, 1] \to \mathscr{A}$ such that $f(0) = a$, $f(1) = I$, and $f(t)$ is an invertible element of $\mathscr{A}$ for each t. The algebra $\mathscr{A}$ has the *connectedness property* if every invertible element of $\mathscr{A}$ may be connected to the identity. We use the term *symmetry* to describe a square root of the identity in a unital Banach algebra $\mathscr{A}$. Such elements have spectrum contained in the set $\{-1, 1\}$ and hence are connected to the identity. In fact, if $\gamma(t)$ is an arc in the complex plane connecting -1 to 1 which does not pass through the origin, then

$$\sigma(t) = \frac{I+S}{2} + \gamma(t)\frac{I-S}{2} \tag{1}$$

is a norm continuous path of invertible elements of $\mathscr{A}$ which connects the symmetry S to the identity I.

The algebra

$$\mathscr{D} = \mathrm{Alg}\,\mathbf{Z} \cap (\mathrm{Alg}\,\mathbf{Z})^*$$

is a von Neumann subalgebra of $\mathrm{Alg}\,\mathbf{Z}$ and since any von Neumann algebra has the connectedness property, we see that any invertible operator in $\mathscr{D}$ can be connected to the identity in $\mathrm{Alg}\,\mathbf{Z}$.

Remark. Let α be a complex number of unit modulus and let $g \in L^\infty(\mathbf{T})$. Let

$$g_\alpha(z) = g(\alpha z), \qquad z \in \mathbf{T},$$

and define a unitary operator $S_\alpha \in \mathscr{D}$ by

$$S_\alpha e_n = \alpha^n e_n,$$

where $e_n(e^{i\theta}) = e^{in\theta}$ is the usual orthonormal basis for $L^2(\mathbf{T})$.

We then have

$$S_\alpha M_g S_\alpha^* = M_{g_\alpha}. \tag{2}$$

Note that by the above remarks, M_g and M_{g_α} belong to the same connectedness class of invertibles in $\mathrm{Alg}\,\mathbf{Z}$.

We now show that M_f can be connected to the identity. Note that $h(z) = -h(-z)$. It follows that we have

$$f(z)f(-z) = 1 \quad \text{for all } z \in \overline{\mathbf{D}}.$$

If $S = S_{-1}$, equation (2) yields,

$$SM_f SM_f = I.$$

Hence both S and SM_f are symmetries in $\operatorname{Alg}\mathbf{Z}$ and

$$M_f = S(SM_f).$$

Therefore M_f can be connected to the identity in $\operatorname{Alg}\mathbf{Z}$. Moreover, equation (1) enables one to obtain an explicit path connecting M_f to the identity.

Question. Let m be a conformal mapping of the disk onto itself and set $g = f \circ m$. Is M_g connected to the identity in $\operatorname{Alg}\mathbf{Z}$? Note that the remark above shows that if m is a rotation, then this is the case.

Remark. Let R be any proper open subset of the complex plane that is simply connected and satisfies $R = -R$. Then $0 \in R$ and if h is any conformal map from the disk onto R with $h(0) = 0$, we have $h(z) = -h(-z)$. (Indeed, the function $g(z) = -h(-z)$ is also a conformal map of the disk onto R. Since $h(0) = g(0)$ and $h'(0) = g'(0)$, the Riemann mapping theorem implies $g = h$.) The argument given above now shows that if we assume that $\{\operatorname{Re}(z) : z \in R\}$ is bounded and set $f = \exp(h)$, then M_f is a product of two symmetries in $\operatorname{Alg}\mathbf{Z}$ and hence is connected to the identity in $\operatorname{Alg}\mathbf{Z}$.

References

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