A NOTE ON THE CONNECTEDNESS PROBLEM FOR NEST ALGEBRAS

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Abstract. It has been conjectured that a certain operator $T$ belonging to the group $\mathcal{F}$ of invertible elements of the algebra $\text{Alg} Z$ of doubly infinite upper-triangular bounded matrices lies outside the connected component of the identity in $\mathcal{F}$. In this note we show that $T$ actually lies inside the connected component of the identity of $\mathcal{F}$.

Let $T$ be the unit circle in the complex plane with normalized Lebesgue measure. For $1 < p < \infty$, let $H^p$ be the usual Hardy space of all functions in $L^p(T)$ that have analytic extensions to the open unit disk $D$. Let $\mathcal{H} = L^2(T)$ and let $\mathcal{B}(\mathcal{H})$ be set of all bounded linear operators on $\mathcal{H}$. Let $W \in \mathcal{B}(\mathcal{H})$ be the shift operator: $(Wf)(e^{i\theta}) = e^{i\theta}f(e^{i\theta})$. In this paper, we consider the nest $\{W^nH^2 : n \in \mathbb{Z}\}$ of subspaces of $L^2(T)$, and its associated nest algebra,

$$\text{Alg} Z = \{T \in \mathcal{B}(\mathcal{H}) : TW^nH^2 \subseteq W^nH^2 \text{ for all } n \in \mathbb{Z}\}.$$

A question which has been unanswered for several years is the following:

Question. Is the group of invertible elements of the Banach algebra $\text{Alg} Z$ connected in the norm topology?

It is frequently conjectured that the answer to this question is no. The reason for conjecturing a negative answer is because of a strong analogy between nest algebras and analytic function theory. We refer the reader to the book by Davidson [1] for details and more background on this question.

For each $f \in L^\infty(T)$, let $M_f \in \mathcal{B}(\mathcal{H})$ be the multiplication operator,

$$M_f \phi = f \phi,$$

$\phi \in L^2(T)$.

Note that for $f \in H^\infty$, we have $M_f \in \text{Alg} Z$. Let $a$ be a positive real number and set

$$h(z) = \frac{ai}{\pi} \log \left( \frac{1 + z}{1 - z} \right).$$

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Then $h$ is a conformal map of the open unit disk onto the unbounded vertical strip $\{z \in \mathbb{C} : -a < \text{Re}(z) < a\}$.

If $f = \exp(h)$ then it is easy to see that both $f$ and $1/f$ are $H^\infty$ functions and moreover, that $f$ is not the exponential of any $H^\infty$ function. Therefore $f$ cannot be connected to the constant function $1$ via a norm continuous path within the group of invertible elements of the Banach algebra $H^\infty$. For this reason, the operator $M_f$ has been suggested as a possible example of an operator which cannot be connected to the identity via a norm continuous path inside the group of invertibles in $\text{Alg}Z$.

The purpose of this note is to show that in fact, $M_f$ may be connected to the identity via a norm continuous path of invertible elements in $\text{Alg}Z$.

Before giving the proof we pause for some terminology and to make a few simple remarks.

Let $\mathscr{A}$ be a unital Banach algebra with unit $I$. Say that an invertible element $a$ of $\mathscr{A}$ may be connected to the identity if there exists a norm continuous function $f: [0, 1] \to \mathscr{A}$ such that $f(0) = a$, $f(1) = I$, and $f(t)$ is an invertible element of $\mathscr{A}$ for each $t$. The algebra $\mathscr{A}$ has the connectedness property if every invertible element of $\mathscr{A}$ may be connected to the identity. We use the term symmetry to describe a square root of the identity in a unital Banach algebra $\mathscr{A}$. Such elements have spectrum contained in the set $\{-1, 1\}$ and hence are connected to the identity. In fact, if $\gamma(t)$ is an arc in the complex plane connecting $-1$ to $1$ which does not pass through the origin, then

$$
\sigma(t) = \frac{I + S}{2} + \gamma(t) \frac{I - S}{2}
$$

is a norm continuous path of invertible elements of $\mathscr{A}$ which connects the symmetry $S$ to the identity $I$.

The algebra

$$
\mathcal{D} = \text{Alg}Z \cap (\text{Alg}Z)^*
$$

is a von Neumann subalgebra of $\text{Alg}Z$ and since any von Neumann algebra has the connectedness property, we see that any invertible operator in $\mathcal{D}$ can be connected to the identity in $\text{Alg}Z$.

**Remark.** Let $\alpha$ be a complex number of unit modulus and let $g \in L^\infty(T)$. Let $g_\alpha(z) = g(\alpha z)$, $z \in T$, and define a unitary operator $S_\alpha \in \mathcal{D}$ by

$$
S_\alpha e_n = \alpha^n e_n,
$$

where $e_n(e^{i\theta}) = e^{i n \theta}$ is the usual orthonormal basis for $L^2(T)$.

We then have

$$
S_\alpha M_g S_\alpha^* = M_{g_\alpha}.
$$

Note that by the above remarks, $M_g$ and $M_{g_\alpha}$ belong to the same connectedness class of invertibles in $\text{Alg}Z$.

We now show that $M_f$ can be connected to the identity. Note that $h(z) = -h(-z)$. It follows that we have

$$
f(z)f(-z) = 1 \quad \text{for all } z \in \mathcal{D}.
$$
If \( S = S_{-1} \), equation (2) yields,
\[
SM_fS = I.
\]
Hence both \( S \) and \( SM_f \) are symmetries in \( \text{Alg} \mathbb{Z} \) and
\[
M_f = S(SM_f).
\]
Therefore \( M_f \) can be connected to the identity in \( \text{Alg} \mathbb{Z} \). Moreover, equation (1) enables one to obtain an explicit path connecting \( M_f \) to the identity.

**Question.** Let \( m \) be a conformal mapping of the disk onto itself and set \( g = f \circ m \). Is \( M_g \) connected to the identity in \( \text{Alg} \mathbb{Z} \)? Note that the remark above shows that if \( m \) is a rotation, then this is the case.

**Remark.** Let \( R \) be any proper open subset of the complex plane that is simply connected and satisfies \( R = -R \). Then \( 0 \in R \) and if \( h \) is any conformal map from the disk onto \( R \) with \( h(0) = 0 \), we have \( h(z) = -h(-z) \). (Indeed, the function \( g(z) = -h(-z) \) is also a conformal map of the disk onto \( R \). Since \( h(0) = g(0) \) and \( h'(0) = g'(0) \), the Riemann mapping theorem implies \( g = h \).) The argument given above now shows that if we assume that \( \{\text{Re}(z) : z \in \mathbb{R}\} \) is bounded and set \( f = \exp(h) \), then \( M_f \) is a product of two symmetries in \( \text{Alg} \mathbb{Z} \) and hence is connected to the identity in \( \text{Alg} \mathbb{Z} \).

**References**


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