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GENERALIZED HYPERGEOMETRIC FUNCTIONS AT UNIT ARGUMENT

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ABSTRACT. The analytic continuation near $z = 1$ of the hypergeometric function ${}_{p+1}F_p(z)$ is obtained for arbitrary $p = 2, 3, \ldots$, including the exceptional cases when the sum of the denominator parameters minus the sum of the numerator parameters is equal to an integer.

1. INTRODUCTION

The behavior near unit argument of the Gaussian hypergeometric function is given by a well-known continuation formula, which, in a form suitable for our purpose, may be written

(1.1)
$$\begin{aligned}\frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(b_1)}\,{}_2F_1\left(\left.\begin{matrix}a_1, a_2\\ b_1\end{matrix}\right| z\right) &= \sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(a_1+n)\Gamma(a_2+n)\Gamma(s_1-n)}{\Gamma(a_1+s_1)\Gamma(a_2+s_1)n!}(1-z)^n\\ &+ (1-z)^{s_1}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(a_1+s_1+n)\Gamma(a_2+s_1+n)\Gamma(-s_1-n)}{\Gamma(a_1+s_1)\Gamma(a_2+s_1)n!}(1-z)^n,\end{aligned}$$

where

$$s_1 = b_1 - a_1 - a_2. \tag{1.2}$$

As it stands, (1.1) is valid if s_1 is not equal to an integer, otherwise it is the starting point from which the required continuation formula follows by an appropriate limiting process.

If $\mathrm{Re}(s_1) > 0$, the hypergeometric function is finite at unit argument and its value given by the Gaussian summation formula

$$\frac{1}{\Gamma(b_1)}\,{}_2F_1\left(\left.\begin{matrix}a_1, a_2\\ b_1\end{matrix}\right| 1\right) = \frac{\Gamma(b_1-a_1-a_2)}{\Gamma(b_1-a_1)\Gamma(b_1-a_2)} = \frac{\Gamma(s_1)}{\Gamma(a_1+s_1)\Gamma(a_2+s_1)}. \tag{1.3}$$

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In this paper we consider hypergeometric series or functions with more parameters [2, 10, 15],

(1.4)

$$ {}_{p+1}F_p\left(\begin{matrix} a_1, a_2, \dots, a_{p+1} \\ b_1, \dots, b_p \end{matrix} \middle| z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_{p+1})_n}{(b_1)_n \cdots (b_p)_n n!} z^n, \qquad |z| < 1, $$

and, besides for a generalization of the Gaussian summation formula (1.3), we are looking for the continuation formula which generalizes (1.1). Its structure was already given in the classic paper by Nørlund [11], but the coefficients were not all known. More recently this problem again received attention in various special cases. The zero-balanced series, so-called when the parameters are such that

$$ (1.5) \qquad s = \sum_{j=1}^{p} b_j - \sum_{j=1}^{p+1} a_j $$

is equal to zero, were considered by Evans and Stanton [9] and Saigo [13], who found the leading terms of the behavior when $z \to 1$ for $p = 2$ and thereby proved a formula of Ramanujan [3, 4, 12, 8]. Now continuation formulas are available for $p = 2$ and s unrestricted [5], and also for $p = 3$ or $p = 4$ when s is not equal to an integer [7]. Finally Saigo and Srivastava [14] obtained the leading terms for arbitrary p in the zero-balanced case.

Here we shall derive the analytic continuation formulas for arbitrary p and unrestricted s.

2. Analytic continuation

We first derive a recurrence relation with respect to p for ${}_{p+1}F_p(z)$. For this purpose it is convenient to introduce

$$ (2.1) \qquad {}_{p+1}G_p(b_p) = \frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_{p+1})}{\Gamma(b_1)\cdots\Gamma(b_p)} {}_{p+1}F_p\left(\begin{matrix} a_1, a_2, \dots, a_{p+1} \\ b_1, \dots, b_p \end{matrix} \middle| z\right), $$

where the dependence on the last denominator parameter, b_p, is indicated for reasons that shall become obvious below. We then have

(2.2)

$$ \begin{aligned} {}_{p+1}G_p(b_p) &= \sum_{n=0}^{\infty} \frac{\Gamma(a_1+n)\cdots\Gamma(a_p+n)\Gamma(a_{p+1}+n)}{\Gamma(b_1+n)\cdots\Gamma(b_{p-2}+n)\Gamma(b_{p-1}+n)\Gamma(b_p+n)n!} z^n \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(a_1+n)\cdots\Gamma(a_p+n)}{\Gamma(b_1+n)\cdots\Gamma(b_{p-2}+n)n!} \\ &\quad \times \frac{1}{\Gamma(b_p+b_{p-1}-a_{p+1}+n)} {}_2F_1\left(\begin{matrix} b_p - a_{p+1}, b_{p-1} - a_{p+1} \\ b_p - b_{p-1} - a_{p+1} + n \end{matrix} \middle| 1\right) z^n, \end{aligned} $$

where by the use of the Gaussian summation formula (1.3) the condition $\mathrm{Re}(a_{p+1}) > 0$ comes in. Writing the ${}_2F_1(1)$ as a series and interchanging the order of the summations we may obtain

$$ (2.3) \quad {}_{p+1}G_p(b_p) = \sum_{m=0}^{\infty} \frac{(b_p - a_{p+1})_m (b_{p-1} - a_{p+1})_m}{m!} {}_pG_{p-1}(b_p + b_{p-1} - a_{p+1} + m) $$

if $\operatorname{Re}(a_{p+1}) > 0$. Repeated use of (2.3) yields

(2.4)

$$\begin{aligned}
{}_{p+1}G_p(b_p) = &\sum_{m_{p-1}=0}^{\infty} \frac{(b_p - a_{p+1})_{m_{p-1}}(b_{p-1} - a_{p+1})_{m_{p-1}}}{m_{p-1}!} \\
&\times \sum_{m_{p-2}=0}^{\infty} \frac{(b_p + b_{p-1} - a_{p+1} - a_p + m_{p-1})_{m_{p-2}}(b_{p-2} - a_p)_{m_{p-2}}}{m_{p-2}!} \\
&\times \sum_{m_{p-3}=0}^{\infty} \frac{(b_p + b_{p-1} + b_{p-2} - a_{p+1} - a_p - a_{p-1} + m_{p-1} + m_{p-2})_{m_{p-3}}(b_{p-3} - a_{p-1})_{m_{p-3}}}{m_{p-3}!} \\
&\times \cdots \\
&\times \sum_{m_1=0}^{\infty} \frac{(b_p + b_{p-1} + \cdots + b_2 - a_{p+1} - a_p - \cdots - a_3 + m_{p-1} + m_{p-2} + \cdots + m_2)_{m_1}(b_1 - a_3)_{m_1}}{m_1!} \\
&\times {}_2G_1(s + a_1 + a_2 + m_{p-1} + m_{p-2} + \cdots + m_1).
\end{aligned}$$

Introducing new indexes of summation,

$$\begin{aligned}
k = k_1 &= m_{p-1} + m_{p-2} + \cdots + m_2 + m_1, \\
k_2 &= m_{p-1} + m_{p-2} + \cdots + m_2, \\
&\vdots \\
k_{p-1} &= m_{p-1},
\end{aligned}$$

and changing the order of the summations accordingly, we have

$$\text{(2.5)} \qquad {}_{p+1}G_p(b_p) = \sum_{k=0}^{\infty} A_k^{(p)}\, {}_2G_1(s + a_1 + a_2 + k),$$

where

$$\text{(2.6)} \qquad A_k^{(2)} = \frac{(b_2 - a_3)_k(b_1 - a_3)_k}{k!},$$

(2.7)

$$A_k^{(3)} = \sum_{k_2=0}^{k} \frac{(b_3 + b_2 - a_4 - a_3 + k_2)_{k-k_2}(b_1 - a_3)_{k-k_2}(b_3 - a_4)_{k_2}(b_2 - a_4)_{k_2}}{(k - k_2)!k_2!},$$

$$\text{(2.8)} \qquad \begin{aligned}
A_k^{(4)} = &\sum_{k_2=0}^{k} \frac{(b_4 + b_3 + b_2 - a_5 - a_4 - a_3 + k_2)_{k-k_2}(b_1 - a_3)_{k-k_2}}{(k - k_2)!} \\
&\times \sum_{k_3=0}^{k_2} \frac{(b_4 + b_3 - a_5 - a_4 + k_3)_{k_2-k_3}(b_2 - a_4)_{k_2-k_3}}{(k_2 - k_3)!} \\
&\times \frac{(b_4 - a_5)_{k_3}(b_3 - a_5)_{k_3}}{k_3!},
\end{aligned}$$

(2.9)

$$A_k^{(p)} = \sum_{k_2=0}^{k} \frac{(b_p + b_{p-1} + \cdots + b_2 - a_{p+1} - a_p - \cdots - a_3 + k_2)_{k-k_2}(b_1 - a_3)_{k-k_2}}{(k-k_2)!}$$
$$\times \sum_{k_3=0}^{k_2} \frac{(b_p + b_{p-1} + \cdots + b_3 - a_{p+1} - a_p - \cdots - a_4 + k_3)_{k_2-k_3}(b_2 - a_4)_{k_2-k_3}}{(k_2-k_3)!}$$
$$\times \cdots$$
$$\times \sum_{k_{p-1}=0}^{k_{p-2}} \frac{(b_p + b_{p-1} - a_{p+1} - a_p + k_{p-1})_{k_{p-2}-k_{p-1}}(b_{p-2} - a_p)_{k_{p-2}-k_{p-1}}}{(k_{p-2}-k_{p-1})!}$$
$$\times \frac{(b_p - a_{p+1})_{k_{p-1}}(b_{p-1} - a_{p+1})_{k_{p-1}}}{k_{p-1}!}$$

or

$$A_k^{(3)} = \frac{(b_3 + b_2 - a_4 - a_3)_k(b_1 - a_3)_k}{k!} \times {}_3F_2\left(\left.\begin{matrix} b_3 - a_4,\, b_2 - a_4,\, -k \\ b_3 + b_2 - a_4 - a_3,\, 1 + a_3 - b_1 - k \end{matrix}\right| 1\right), \tag{2.10}$$

$$A_k^{(4)} = \frac{(b_4 + b_3 + b_2 - a_5 - a_4 - a_3)_k(b_1 - a_3)_k}{k!} \times \sum_{l=0}^{k} \frac{(b_4 + b_3 - a_5 - a_4)_l(b_2 - a_4)_l(-k)_l}{(b_4 + b_3 + b_2 - a_5 - a_4 - a_3)_l(1 + a_3 - b_1 - k)_l\, l!} \times {}_3F_2\left(\left.\begin{matrix} b_4 - a_5,\, b_3 - a_5,\, -l \\ b_4 + b_3 - a_5 - a_4,\, 1 + a_4 - b_2 - l \end{matrix}\right| 1\right), \tag{2.11}$$

(2.12)

$$A_k^{(p)} = \frac{(b_p + b_{p-1} + \cdots + b_2 - a_{p+1} - a_p - \cdots - a_3)_k(b_1 - a_3)_k}{k!}$$
$$\times \sum_{k_2=0}^{k} \frac{(-k)_{k_2}}{(b_p + b_{p-1} + \cdots + b_2 - a_{p+1} - a_p - \cdots - a_3)_{k_2}(1 + a_3 - b_1 - k)_{k_2}}$$
$$\times \frac{(b_p + b_{p-1} + \cdots + b_3 - a_{p+1} - a_p - \cdots - a_4)_{k_2}(b_2 - a_4)_{k_2}}{k_2!}$$
$$\times \sum_{k_3=0}^{k_2} \frac{(-k_2)_{k_3}}{(b_p + b_{p-1} + \cdots + b_3 - a_{p+1} - a_p - \cdots - a_4)_{k_3}(1 + a_4 - b_2 - k_2)_{k_3}}$$
$$\times \cdots$$

$$\times \frac{(b_p + b_{p-1} - a_{p+1} - a_p)_{k_{p-2}} (b_{p-2} - a_p)_{k_{p-2}}}{k_{p-2}!}$$

$$\times \sum_{k_{p-1}=0}^{k_{p-2}} \frac{(-k_{p-2})_{k_{p-1}}}{(b_p + b_{p-1} - a_{p+1} - a_p)_{k_{p-1}} (1 + a_p - b_{p-2} - k_{p-2})_{k_{p-1}}}$$

$$\times \frac{(b_p - a_{p+1})_{k_{p-1}} (b_{p-1} - a_{p+1})_{k_{p-1}}}{k_{p-1}!}$$

by means of the identity

$$\frac{(B)_{m-k}}{(m-k)!} = \frac{(B)_m (-m)_k}{m!(1-B-m)_k}. \tag{2.13}$$

We first consider the case when $z = 1$. Evaluating the ${}_2G_1$ in (2.5) by means of (1.3) with $b_1 = s + a_1 + a_2 + k$, we may prove the following result.

Theorem 1. *If, with* s *according to* (1.5), $\mathrm{Re}(s) > 0$, *the generalization of the Gaussian summation formula for* $p = 2, 3, \ldots$ *is*

$$\begin{aligned} &\frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_{p+1})}{\Gamma(b_1)\cdots\Gamma(b_p)} {}_{p+1}F_p\left(\left. \begin{matrix} a_1, a_2, \ldots, a_{p+1} \\ b_1, \ldots, b_p \end{matrix} \right| 1 \right) \\ &= \frac{\Gamma(a_1)\Gamma(a_2)\Gamma(s)}{\Gamma(a_1+s)\Gamma(a_2+s)} \sum_{k=0}^{\infty} \frac{(s)_k}{(a_1+s)_k (a_2+s)_k} A_k^{(p)}, \end{aligned} \tag{2.14}$$

where the $A_k^{(p)}$ *are given by (2.6)–(2.12) and the series converges for* $\mathrm{Re}(a_3) > 0 \wedge \cdots \wedge \mathrm{Re}(a_{p+1}) > 0$.

For $p = 2$ Theorem 1 reduces to a known formula [2, 10].

On the other hand, by substituting for the ${}_2G_1$ in (2.5) its analytic continuation according to (1.1) with $b_1 = s + a_1 + a_2 + k$, we may prove the following continuation formula.

Theorem 2. *If* s *according to* (1.5) *is not equal to an integer, the coefficients in the continuation formula*

$$\begin{aligned} &\frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_{p+1})}{\Gamma(b_1)\cdots\Gamma(b_p)} {}_{p+1}F_p\left(\left. \begin{matrix} a_1, a_2, \ldots, a_{p+1} \\ b_1, \ldots, b_p \end{matrix} \right| z \right) \\ &= \sum_{n=0}^{\infty} g_n(0)(1-z)^n + (1-z)^s \sum_{n=0}^{\infty} g_n(s)(1-z)^n \end{aligned} \tag{2.15}$$

for $|1-z|<1$, $|\arg(1-z)|<\pi$, *and* $p=2, 3, \ldots$ *are*

$$\begin{aligned} g_n(r) &= (-1)^n \frac{\Gamma(a_1+r+n)\Gamma(a_2+r+n)\Gamma(s-2r-n)}{\Gamma(a_1+s)\Gamma(a_2+s)n!} \\ &\times \sum_{k=0}^{\infty} \frac{(s-r-n)_k}{(a_1+s)_k(a_2+s)_k} A_k^{(p)}, \end{aligned} \tag{2.16}$$

where the $A_k^{(p)}$ *are given by* (2.6)–(2.12) *and, while the series terminates when* $r=s$, *the condition* $\operatorname{Re}(a_3+n)>0 \wedge \cdots \wedge \operatorname{Re}(a_{p+1}+n)>0$ *is required for convergence of the series* (2.16) *in the case of* $r=0$.

3. The exceptional cases

There are now two ways to treat the exceptional cases when s is equal to an integer.

We may start from Theorem 2 with $-\varepsilon$ added to each of the parameters and, as a consequence, s replaced by $s+\varepsilon$. Then, on the assumption that s is an integer, either ≤ 0 or ≥ 0, the limit $\varepsilon \to 0$ is performed. Since all the $A_k^{(p)}$ are not influenced by the replacements and so remain independent of ε, the whole procedure is here not more complicated than in the case of $p=2$ and leads to results which formally resemble those of [5].

Alternatively, we may start from (2.5) and substitute for the ${}_2G_1$ its analytic continuation. Each of the required continuation formulas, which follows from (1.1) by an appropriate limiting process, is available from (15.3.10)–(15.3.12) of [1], for instance. It was this procedure, essentially, which Saigo [13] used in order to obtain the leading terms for $p=2$ in the zero-balanced case. In our more general setting this procedure is still simple for $s \geq 0$, but is less convenient for $s<0$.

Any of these two ways leads to the following results.

Theorem 3. *If* s *according to* (1.5) *is equal to an integer* $t \geq 0$, *then the coefficients in the continuation formula*

$$\begin{aligned} &\frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_{p+1})}{\Gamma(b_1)\cdots\Gamma(b_p)} {}_{p+1}F_p\left(\begin{matrix} a_1, a_2, \ldots, a_{p+1} \\ b_1, \ldots, b_p \end{matrix} \middle| z \right) \\ &= \sum_{n=0}^{t-1} l_n(1-z)^n + (1-z)^t \sum_{n=0}^{\infty} [w_n + q_n \ln(1-z)](1-z)^n \end{aligned} \tag{3.1}$$

for $|1-z|<1$, $|\arg(1-z)|<\pi$, *and* $p=2, 3, \ldots$ *are*

$$l_n = (-1)^n \frac{\Gamma(a_1+n)\Gamma(a_2+n)(t-n-1)!}{\Gamma(a_1+t)\Gamma(a_2+t)n!} \sum_{k=0}^{\infty} \frac{(t-n)_k}{(a_1+t)_k(a_2+t)_k} A_k^{(p)}, \tag{3.2}$$

$$
\begin{aligned}
w_n + q_n \ln(1-z) &= (-1)^t \frac{(a_1+t)_n (a_2+t)_n}{(t+n)!n!} \\
&\times \left\{ \sum_{k=0}^{n} \frac{(-n)_k}{(a_1+t)_k (a_2+t)_k} A_k^{(p)} \times \{\psi(1+n-k) + \psi(1+t+n) \right. \\
&\qquad - \psi(a_1+t+n) - \psi(a_2+t+n) - \ln(1-z)\} \\
&\qquad \left. + (-1)^n n! \sum_{k=n+1}^{\infty} \frac{(k-n-1)!}{(a_1+t)_k (a_2+t)_k} A_k^{(p)} \right\},
\end{aligned} \tag{3.3}
$$

where the $A_k^{(p)}$ *are given by* (2.6)–(2.12) *and the conditions* $\operatorname{Re}(a_3+n) > 0 \wedge \cdots \wedge \operatorname{Re}(a_{p+1}+n) > 0$ *or* $\operatorname{Re}(a_3+n+t) > 0 \wedge \cdots \wedge \operatorname{Re}(a_{p+1}+n+t) > 0$ *are required for convergence of the infinite series in* (3.2) *or* (3.3), *respectively.*

Theorem 4. *If* s *according to* (1.5) *is equal to an integer* $-t \le 0$, *then the coefficients in the continuation formula*

$$
\begin{aligned}
&\frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_{p+1})}{\Gamma(b_1)\cdots\Gamma(b_p)} {}_{p+1}F_p \left(\left. \begin{matrix} a_1, a_2, \dots, a_{p+1} \\ b_1, \dots, b_p \end{matrix} \right| z \right) \\
&\quad = (1-z)^{-t} \sum_{n=0}^{t-1} h_n (1-z)^n + \sum_{n=0}^{\infty} [u_n + v_n \ln(1-z)](1-z)^n
\end{aligned} \tag{3.5}
$$

for $|1-z| < 1$, $|\arg(1-z)| < \pi$, *and* $p = 2, 3, \dots$ *are*

$$
h_n = (-1)^n \frac{(a_1-t)_n (a_2-t)_n (t-n-1)!}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{(a_1-t)_k (a_2-t)_k} A_k^{(p)}, \tag{3.6}
$$

$$
\begin{aligned}
u_n + v_n \ln(1-z) &= (-1)^t \frac{(a_1-t)_{t+n} (a_2-t)_{t+n}}{n!(t+n)!} \\
&\times \left\{ \sum_{k=0}^{t+n} \frac{(-t-n)_k}{(a_1-t)_k (a_2-t)_k} A_k^{(p)} \{\psi(1+t+n-k) + \psi(1+n) \right. \\
&- \psi(a_1+n) - \psi(a_2+n) - \ln(1-z)\} \\
&\qquad \left. + (-1)^{t+n} (t+n)! \sum_{k=t+n+1}^{\infty} \frac{(k-t-n-1)!}{(a_1-t)_k (a_2-t)_k} A_k^{(p)} \right\},
\end{aligned} \tag{3.7}
$$

where the $A_k^{(p)}$ *are given by* (2.6)–(2.12) *and the condition* $\operatorname{Re}(a_3+n) > 0 \wedge \cdots \wedge \operatorname{Re}(a_{p+1}+n) > 0$ *is required for convergence of the infinite series in* (3.7).

When $t = 0$, the empty sums in (3.1) or (3.5) have to be interpreted as 0.

4. Other representations

There are, for $p = 3, 4, \dots$, other representations of the $A_k^{(p)}$ that are nontrivially different from those of this work. By comparison with the results of [6] we have the alternative expressions

$$
\begin{aligned}
A_k^{(3)} &= \frac{(b_1+b_3-a_3-a_4)_k (b_2+b_3-a_3-a_4)_k}{k!} \\
&\quad \times {}_3F_2 \left(\left. \begin{matrix} b_3-a_3, b_3-a_4, -k \\ b_1+b_3-a_3-a_4, b_2+b_3-a_3-a_4 \end{matrix} \right| 1 \right),
\end{aligned} \tag{4.1}
$$

(4.2)

$$A_k^{(4)} = \frac{(b_1+b_3+b_4-a_3-a_4-a_5)_k(b_2+b_3+b_4-a_3-a_4-a_5)_k}{k!}$$
$$\times \sum_{l=0}^{k} \frac{(b_3+b_4-a_3-a_5)_l(b_3+b_4-a_4-a_5)_l(-k)_l}{(b_1+b_3+b_4-a_3-a_4-a_5)_l(b_2+b_3+b_4-a_3-a_4-a_5)_l l!}$$
$$\times {}_3F_2\left(\begin{matrix} b_3-a_5\,,\ b_4-a_5\,,\ -l \\ b_3+b_4-a_3-a_5\,,\ b_3+b_4-a_4-a_5 \end{matrix}\middle| 1\right),$$

which may be used in place of (2.10) or (2.11), respectively.

5. The zero-balanced case

As mentioned in the introduction, the zero-balanced case has received considerable attention, and therefore we shall discuss it in more detail. It is covered by the following corollary.

Corollary 1. *If* s *according to* (1.5) *is equal to zero, then the continuation formula*

(5.1)

$$\frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_{p+1})}{\Gamma(b_1)\cdots\Gamma(b_p)}\,{}_{p+1}F_p\left(\begin{matrix} a_1\,,\ a_2\,,\ \dots\,,\ a_{p+1} \\ b_1\,,\ \dots\,,\ b_p \end{matrix}\middle| z\right)$$
$$= \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n}{n!n!}\left\{\sum_{k=0}^{n} \frac{(-n)_k}{(a_1)_k(a_2)_k} A_k^{(p)} \{\psi(1+n-k)+\psi(1+n) - \psi(a_2+n) - \ln(1-z)\} + (-1)^n(n)! \sum_{k=n+1}^{\infty} \frac{(k-n-1)!}{(a_1)_k(a_2)_k} A_k^{(p)}\right\}(1-z)^n$$

holds for $|1-z|<1$, $|\arg(1-z)|<\pi$, *and* $p=2, 3, \dots$, *where the* $A_k^{(p)}$ *are given by* (2.6)–(2.12) *and the condition* $\mathrm{Re}(a_3+n)>0 \wedge \cdots \wedge \mathrm{Re}(a_{p+1}+n)>0$ *is required for convergence of the infinite series.*

So the behavior when $z \to 1$ of the zero-balanced series (5.1) is

(5.2) $\quad [2\psi(1)-\psi(a_1)-\psi(a_2)+B][1+O(1-z)] - \ln(1-z)[1+O(1-z)]$

where, if the condition $\mathrm{Re}(a_j)>0$ for $j=3, 4, \dots, p+1$ is satisfied,

$$B = \sum_{k=1}^{\infty} \frac{(k-1)!}{(a_1)_k(a_2)_k} A_k^{(p)}. \tag{5.3}$$

If $p \geq 3$, different representations follow for B according as (2.7)–(2.9), or (2.10)–(2.12), or (4.1)–(4.2) is used for the $A_k^{(p)}$. Still another representation is due to Saigo and Srivastava [14].

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